Lecture 28-29
FM- Frequency Modulation
PM - Phase Modulation

EE445-10

FM and PM

for FM:
\[ \theta(t) = D_f \int_{-\infty}^{t} m(\sigma) \, d\sigma \]  
(5–36)

Relationship between \( m_f(t) \) and \( m_p(t) \):
\[ m_f(t) = \frac{D_f}{D_p} \left[ \frac{d m_p(t)}{d t} \right] \]  
(5–37)

where the subscripts \( f \) and \( p \) denote frequency and phase, respectively. Similarly, if we have an FM signal modulated by \( m_f(t) \), the corresponding phase modulation on this signal is
\[ m_p(t) = \frac{D_p}{D_f} \int_{-\infty}^{t} m_f(\sigma) \, d\sigma \]  
(5–38)

Figure 5–8 Angle modulator circuits. RFC = radio-frequency choke.

\( D_p \) is the phase sensitivity or phase modulation constant.
**FM and PM**

**DEFINITION.** If a bandpass signal is represented by

\[ s(t) = R(t) \cos(\phi(t)) \]

where \( \phi(t) = \omega_c t + \theta(t) \), then the **instantaneous frequency** (hertz) of \( s(t) \) is \([\text{Brooshax, 1992}])\]

\[ \dot{\phi}(t) = \frac{1}{2\pi} \omega_c(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} \]

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**Figure 5-9**

**FM and PM differences**

- **PM:**
  \[ \theta(t) = D_p m(t) \Rightarrow \text{phase is proportional to } m(t) \]

- **FM:**
  \[ \theta(t) = D_f \int_{-\infty}^{t} m(\alpha) d\alpha \]
  \[ f_i(t) - f_c = D_f m(t) \Rightarrow \text{instantaneous frequency deviation from the carrier is proportional to } m(t) \]

**Modulation Constants**

\[ D_p = K_p \Rightarrow \text{radians/volt} \]
\[ D_f = K_f \Rightarrow \text{Hz/volt} \]
FM and PM Signals

Maximum phase deviation in PM:

$$\Delta \phi_{\text{MAX}} = k_p \max[|m(t)|].$$

Maximum frequency deviation in FM:

$$\Delta f_{\text{MAX}} = k_f \max[|m(t)|].$$

Example

Let $m(t) = a \cos(2 \pi f_m t)$

For PM

$$\phi(t) = k_p m(t) = k_p a \cos(2 \pi f_m t).$$

For FM

$$\phi(t) = 2 \pi k_f \int_{-\infty}^{t} m(\tau) \, d\tau = \frac{k_f a}{f_m} \sin(2 \pi f_m t).$$

$$u(t) = \begin{cases} A_c \cos \left(2 \pi f_c t + k_p a \cos(2 \pi f_m t)\right), & \text{PM} \\ A_c \cos \left(2 \pi f_c t + \frac{k_f a}{f_m} \sin(2 \pi f_m t)\right), & \text{FM} \end{cases}.$$
Example

Define the modulation indices:

\[ \beta_p = k_p a \]
\[ \beta_f = \frac{k_f a}{f_m} \]
\[ \beta_p = k_p \max||m(t)|| \]
\[ \beta_f = \frac{k_f \max||m(t)||}{W} \]
\[ \beta_p = \Delta \phi_{\text{max}} \]
\[ \beta_f = \frac{\Delta f_{\text{max}}}{W} \]

Spectrum Characteristics of FM

- FM/PM is exponential modulation

Let \( \phi(t) = \beta \sin(2\pi f_m t) \)

\[ u(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \]
\[ = \text{Re}(A_c e^{j(2\pi f_c t + \beta \sin(2\pi f_m t))}) \]

\( u(t) \) is periodic in \( f_m \)
we may therefore use the Fourier series

Sine Wave Example

Then

\[ u(t) = \begin{cases} 
A_c \cos(2\pi f_c t + \beta_p \cos(2\pi f_m t)) , & \text{PM} \\
A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_m t)) , & \text{FM} 
\end{cases} \]

Spectrum Characteristics of FM

- FM/PM is exponential modulation

\[ c(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \]
\[ = \text{Re}(A_c e^{j(2\pi f_c t + \beta \sin(2\pi f_m t))}) \]

\( c(t) \) is periodic in \( f_m \)
we may therefore use the Fourier series
Spectrum Characteristics with Sinusoidal Modulation

\[ e^{j\beta \sin 2\pi f_m t} \cdot \]

\( u(t) \) is periodic in \( f_m \), we may therefore use the Fourier series

\[
\begin{align*}
\alpha_n &= f_m \int_{fm} \frac{1}{f_m} e^{j\beta \sin 2\pi f_m t} e^{-j2\pi f_m t} dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{j\beta(\sin u - u)} du.
\end{align*}
\]

\[ J_n Bessel Function \]

\[ J_n(\beta) \approx \frac{\beta^n}{2^n n!} \cdot \]

\[ J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases} \]

Figure 5–11 Magnitude spectra for FM or PM with sinusoidal modulation for various modulation indices.
Lecture 29
FM - Frequency Modulation
PM - Phase Modulation
(continued)

Narrowband FM
• Only the $J_0$ and $J_1$ terms are significant
• Same Bandwidth as AM
• Using Euler's identity, and $\phi(t) \ll 1$:

$$v(t) = A_c \cos 2\pi f_c t \cos \phi(t) - A_c \sin 2\pi f_c t \sin \phi(t)$$
$$\approx A_c \cos 2\pi f_c t - A_c \phi(t) \sin 2\pi f_c t,$$

Notice the sidebands are "sin", not "cos" as in AM

Wideband FM from Narrowband FM
• The Output Carrier frequency = $n \times f_c$
• The output modulation index = $n \times \beta_c$
• The output bandwidth increases according to Carson's Rule
Effective Bandwidth- Carson’s Rule for Sine Wave Modulation

\[ B_c = 2(\beta + 1)f_m, \quad \text{Where } \beta \text{ is the modulation index} \]

\[ m(t) = a \cos(2\pi f_m t). \]

\[ B_c = \begin{cases} 2(k_P a + 1)f_m, & \text{PM} \\ 2k_P f_m^0 + 1 & \text{FM} \end{cases} \]

• Notice for FM, if \( k_P a \ll f_m \), increasing \( f_m \) does not increase \( B_c \) much.
• \( B_c \) is linear with \( f_m \) for PM.

Figure 5–11
Magnitude spectra for FM or PM with sinusoidal modulation for various modulation indexes.
When \( m(t) \) is a sum of sine waves

Consider now the case where \( m(t) \) is the sum of \( K \) separate sine waves. That is, let

\[
m(t) = \sum_{k=1}^{K} C_k \cos (\omega_k t + \theta_k)
\]

(2.4.14)

where \( C_k, \omega_k, \) and \( \theta_k \) are the corresponding deviations, frequency, and phase angles. We then have

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**Sideband Power**

- **Signal Amplitude:** \( A_c = 1 \text{V} \)
- **Modulating frequency:** \( f_m = 1 \text{kHz} \)
- **Carrier peak deviation:** \( \Delta f = 2.4 \text{kHz} \)
- **Modulation index:** \( \beta = \frac{\Delta f}{f_m} = 2.4 \)
- **Reference equation:**
  \[
  x(t) = \sum_{n=-\infty}^{\infty} \left[ A_n \cos (n \omega_m t) \right] \cos \left[ \omega_c t + \sum_{k=1}^{K} C_k (\omega_k t + \theta_k) \right]
  \]
  \[
  \sum \beta
  \]
- **Power in the signal:** \( P_c = \frac{A_c^2}{2} = 0.5 \text{W} \)
- **Carsons rule bandwidth:** \( BW = 2 (\beta + 1) f_m = 6.8 \times 10^3 \text{Hz} \)
- **Order of significant sidebands predicted by Carsons rule:** \( n = \alpha \frac{\Delta f}{f_m} + 1 = 3 \)
- **Power as a function of number of sidebands:**
  \[
  P_{\text{sum}(k)} = \sum_{n=\alpha}^{\infty} \frac{[A_n \cos (n \Delta f)]^2}{2} = \frac{P_c}{100 - 99.118}
  \]
- **Percent of power predicted by Carsons rule:**
  \[
  \frac{P_{\text{sum}(k)}}{P_c} = 99.118
  \]

---

When \( m(t) \) is a sum of sine waves

\[
\begin{align*}
\phi(t) &= A \cos \left[ \omega_c t + \sum_{j=1}^{K} \beta_j \sin (\omega_j t + \theta_j) \right] \\
\phi(t) &= A \sum_{k=1}^{K} \sum_{m=1}^{\infty} \left[ 1 \cos (\omega_c t + \sum_{j=1}^{K} \beta_j (\omega_j t + \theta_j) + \phi) \right]
\end{align*}
\]

(2.4.16)

The preceding equation represents the general expression for the FM carrier modulated by \( K \) sinusoids. Note that it corresponds to a collection of harmonic frequencies at all the sidebands \( \sum_{j=1}^{K} \beta_j \omega_j \), where all combinations of integers for the \( (\beta_j) \) must be considered. Each such combination \( (k_1, k_2, \ldots, k_K) \) yields a different sinusoid, each with its own phase, \( \sum_{j=1}^{K} (\beta_j \omega_j + \phi) \) and its own amplitude \( (A \prod_{j=1}^{K} I_j(B_j)) \). In particular, we note that the carrier component at \( \omega_c \) corresponds to \( k_1 = k_2 = \cdots = k_K = 0 \) and has amplitude \( (A \prod_{j=1}^{K} I_j(B_j)) \), whereas the frequency component at frequency \( (\omega + \omega_c) \) corresponds to \( k_1 = \alpha, k_2 = 0, \ldots, k_K = 0 \) and has amplitude \( (A \prod_{j=1}^{K} I_j(B_j)) \). We also note that the component at \( (\omega + \omega_c) \) contains the exact phase angle of the \( j \)th sine wave in (2.4.14) added to that of the carrier.
### Sideband Power

#### Formulas

- **Sideband Power**
  
  \[ k := 0 \ldots 10 \]
  
  \[ J(k) := J_0(k, \beta) \]
  
  \[ P_k := (J_k)^2 \]

- **Power Equation**
  
  \[ \beta := 0.24 \]
  
  \[ n := 3 \]

#### Table

<table>
<thead>
<tr>
<th>( J )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>1</td>
<td>0.32</td>
</tr>
<tr>
<td>2</td>
<td>0.431</td>
</tr>
<tr>
<td>3</td>
<td>0.139</td>
</tr>
<tr>
<td>4</td>
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<td>5</td>
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<td>7</td>
<td>3.907</td>
</tr>
<tr>
<td>8</td>
<td>4.707</td>
</tr>
<tr>
<td>9</td>
<td>1.036</td>
</tr>
<tr>
<td>10</td>
<td>1.946</td>
</tr>
</tbody>
</table>

#### Calculations

- **Average Sideband Power**
  
  \[ P_0 = \frac{1}{n} \sum_{j=1}^{n} P_j = 0.991 \]

- **Sideband Calculation**
  
  \[ J := J_0(n, \beta) \]

#### Results

- **Sideband Power Table**

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \beta )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>1</td>
</tr>
</tbody>
</table>

#### Modulation Index

- **Formula**
  
  \[ \xi := 0.06^2 \]

- **Modulation Frequency**
  
  \[ \xi := 0.05^2 \]

#### FM/PM Modulation

- **Equation**
  
  \[ M = \frac{\Delta f}{f_m} \]

- **Number of Sidebands**
  
  \[ n := \text{round}(M + 1) \]

- **Bandwidth**
  
  \[ 2n \text{ is the number of significant sidebands per Carson's rule} \]

- **Sideband Formula**
  
  \[ S(\theta) = A_c \left[ (\text{OR}) M \cdot \left| \xi_j \right|^2 \right] + \sum_{k=1}^{n} \left[ M \cdot \xi_j \cdot \left| \xi_j + k \frac{\Delta f}{f_m} \right| \right] \]

- **Sideband Calculation**
  
  \[ R = A_c \left[ (\text{OR}) M \cdot \left| \xi_j \right|^2 \right] + \sum_{k=1}^{n} \left[ M \cdot \xi_j \cdot \left| \xi_j + k \frac{\Delta f}{f_m} \right| \right] \]

#### Parameters

- **Modulating Frequency**
  
  \[ \text{Modulating frequency - single sinewave} \]

- **FM/PM Modulation Index**
  
  \[ M = \frac{\Delta f}{f_m} \]

- **Modulation Index**
  
  \[ \xi = 0.06^2 \]

- **Modulation Factor**
  
  \[ R = A_c \left[ (\text{OR}) M \cdot \left| \xi_j \right|^2 \right] + \sum_{k=1}^{n} \left[ M \cdot \xi_j \cdot \left| \xi_j + k \frac{\Delta f}{f_m} \right| \right] \]

#### Note

- **Filename**
  
  `fmsidebands.mcd`

- **Date**
  
  `09/01/04`

- **Last Edit Date**
  
  `02/27/07`
M=0.4, Sideband Level = M/2 for Narrowband FM

M=2.4, Carrier Null

M=0.9, Sideband Level = M/2 for Narrowband FM

M=3.8, first sideband null
M = 5.1, second sideband null

Power vs BW, M = 0.9

Power vs BW, M = 0.1

Power vs BW, M = 2.4
AM vs FM

- **FM capture effect:** A phenomenon, associated with FM reception, in which only the stronger of two signals at or near the same frequency will be demodulated.
  - The complete suppression of the weaker signal occurs at the receiver limiter, where it is treated as noise and rejected.
  - When both signals are nearly equal in strength, or are fading independently, the receiver may switch from one to the other.
- **Bandwidth:** $B_{AM} = 2 \times f_m$, $B_{FM} \geq 2 \times f_m$ use Carson's Rule
- The Receiver IF amplifier is change to a Limiting Amplifier for FM
  - FM rejects amplitude noise such as lightening and man made noise.
- The FM demodulator may be a PLL, Ratio Detector, Foster Sealy Discriminator, or slope detector.

\[
H_p(f) = \frac{1}{1 + \mu \frac{2f_c}{f_m}}
\]

where $f_c = \frac{1}{2\pi R}$ and $f_m = \frac{1}{2\pi R}$. 

\[
H_p(f) = \frac{1}{1 + \mu \frac{2f_c}{f_m}} = \frac{1}{1 + \mu \frac{f_m}{f_c}}
\]