EELE503

Modern filter design

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Filter Design - Introduction

- A filter will modify the magnitude or phase of a signal to produce a desired frequency response or time response.
- One way to classify ideal filters is by frequency response
  - Lowpass $|H(0<f<B)|=1$, $|H(f)|=0$ elsewhere
  - Highpass $|H(f>B)|=1$, $|H(f)|=0$ elsewhere
  - Bandpass $|H(f1<f<f2)=1$, $|H(f)|=0$ elsewhere $B=f2-f1$
  - Bandstop $|H(f1<f<f2| =0$, $|H(f)|=1$ elsewhere $B=f2=f1$
    also called a band-reject filter
  - Allpass $|H(f)|=1$, $\arg H(f)=\theta(f)$

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Filter Design - Introduction

- An ideal filter will pass desired frequencies with no loss or phase distortion, and provide infinite attenuation to unwanted frequencies.
- It may be shown that an ideal rectangular filter response would require an infinite number of poles to realize.
- Modern analog filter design results in an approximation to the desired ideal response.

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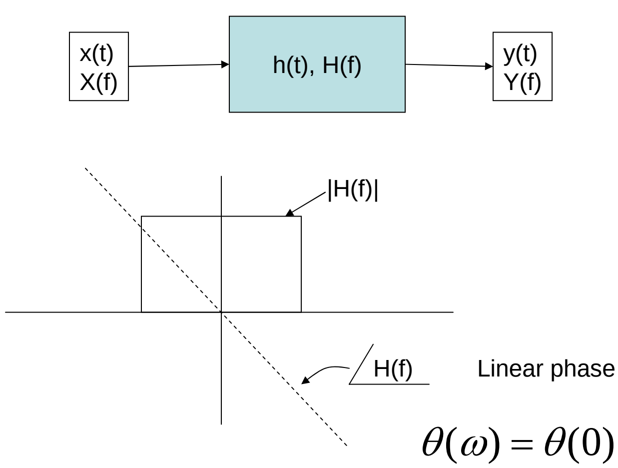
Ideal Filter



$$\theta(\omega) = \theta(0) - \tau_d \bullet \omega$$

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Ideal Filter-Magnitude, Phase/Delay

For a transfer function H(s), at real frequencies, with s=jω,

$$H(j2\pi f) = |H(j2\pi f)| e^{j\theta(j2\pi f)} = G(\omega) e^{j\theta(\omega)}$$

Where G(ω) and θ(ω) are the gain and phase components.

Phase Delay Pd(ω) is defined as: $$Pd(\omega) = \frac{-\theta(\omega)}{\omega}$$

Group Delay $\tau_d(\omega)$ is defined as: $$\tau_d(\omega) = -\frac{\partial\theta(\omega)}{\partial\omega}$$

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Ideal Filter-Magnitude, Phase/Delay

Linear Phase: $$\theta(\omega) = \theta(0) - \tau_d \bullet \omega$$

Linear Phase Distortion: $$\theta(\omega) = \theta(0) - \tau_d \omega - \tau_2 \omega^2 - \ldots$$

The τ2 term is called the parabolic group delay distortion and has units of sec^2

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Ideal Filter-Magnitude, Phase/Delay

- Both Pd(ω) and $\tau_d(\omega)$ are functions of frequency
- Phase delay Pd(ω) is the absolute delay and is usually of little significance
- Group Delay $\tau_d(\omega)$ is used as the criterion to evaluate phase nonlinearity. Group Delay is constant for all frequencies in the passband of an ideal filter.

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Ideal Filter-Magnitude, Phase/Delay

- Linear phase variation with frequency (over a band of frequencies) implies a constant Group Delay –no phase distortion in that band of frequencies
- In order to preserve the integrity of a pulse x(t), it is mandatory that the Group Delay of the system be constant up to the maximum frequency component of the pulse. This implies equal time delay for all frequencies of interest.

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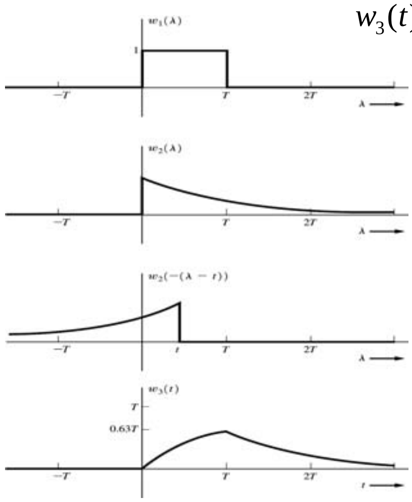
Convolution of a rectangle and an exponential.

$$w_3(t) = w_1(t) * w_2(t) \equiv \int_{-\infty}^{\infty} w_1(\lambda) w_2(t-\lambda) d\lambda$$



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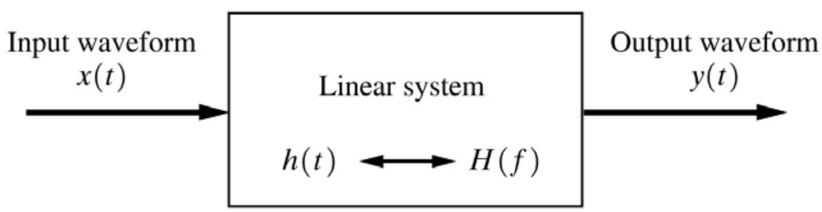
Power Signal Through a Filter:



Some descriptions for the input		Some descriptions for the output
$X(f)$	"Voltage" spectrum	$Y(f)$
$R_x(\tau)$	Autocorrelation function	$R_y(\tau)$
$\mathcal{P}_x(f)$	Power spectral density	$\mathcal{P}_y(f)$

Power Signal Through a Filter:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

time average or power autocorr:

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \left[\int_{-\infty}^{\infty} h(u)x(t-u)\,du\right]\left[\int_{-\infty}^{\infty} h^*(v)x^*(t-\tau-v)\,dv\right] dt.$$

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} y(t)y^*(t-\tau)\,dt.$$

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Periodic Signal Through a Filter: Time Average Autocorrelation

$$R_x(\tau) = Lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)dt = Lim_{k\to\infty} \frac{1}{kT_o} \int_{-\frac{kT_o}{2}}^{\frac{kT_o}{2}} x(t)x^*(t-\tau)dt$$

$$= Lim_{k\to\infty} \frac{k}{kT_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t-\tau)dt$$

$$R_x(\tau) = \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t-\tau)dt$$ for a periodic signal

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Discreet Time Average Autocorrelation

$$R_x(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{+\infty} x_n x_m^* e^{j2\pi\frac{m}{T_0}\tau} e^{j2\pi\frac{n-m}{T_0}t}\,dt.$$

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} e^{j2\pi\frac{n-m}{T_0}t}\,dt = \delta_{mn},$$

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi\frac{n}{T_0}\tau}.$$

$$\mathcal{S}_x(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right).$$

$$P_x = \sum_{n=-\infty}^{\infty} |x_n|^2.$$

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Power Signal Through a Filter:

By making a change of variables $w = t - u$ and changing the order of integration, we obtain

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)$$

$$\times \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}-u}^{\frac{T}{2}+u}\left[x(w)x^*(u+w-\tau-v)\,dw\right]du\,dv$$

$$\stackrel{\text{a}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau+v-u)h(u)h^*(v)\,du\,dv$$

$$\stackrel{\text{b}}{=} \int_{-\infty}^{\infty} [R_x(\tau+v)\star h(\tau+v)]\,h^*(v)\,dv$$

$$\boxed{\stackrel{\text{c}}{=} R_x(\tau)\star h(\tau)\star h^*(-\tau),}$$

$$\mathcal{S}_y(f) = \mathcal{S}_x(f)H(f)H^*(f)$$

$$\boxed{= \mathcal{S}_x(f)|H(f)|^2.}$$

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Discrete Time Average Autocorrelation Through a LTI System

$$\mathcal{S}_y(f) = |H(f)|^2 \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right)$$
$$= \sum_{n=-\infty}^{\infty} |x_n|^2 \left|H\left(\frac{n}{T_0}\right)\right|^2 \delta\left(f - \frac{n}{T_0}\right)$$

$$P_y = \sum_{n=-\infty}^{\infty} |x_n|^2 \left|H\left(\frac{n}{T_0}\right)\right|^2.$$

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Power Autocorrelation: R(0)= power

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)x^*(t-\tau)\,dt.$$

$$\mathcal{S}_x(f) = \mathscr{F}[R_x(\tau)].$$

$$P_x = R_x(0) = \int_{-\infty}^{\infty} \mathcal{S}_x(f)\,df.$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 dt = R_x(0).$$

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Filters-Applications

- Modify the frequency spectrum of a signal
  - remove out of band distortion
  - Reduce the magnitude of unwanted signals, example 60 Hz hum
- reduce noise power by reducing bandwidth
- Waveform Shaping: $y(t)=x(t)*h(t)$
- Matched signal detection

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http://en.wikipedia.org/wiki/Butterworth_filter

Linear analog electronic filters
Butterworth filter
Chebyshev filter
Elliptic (Cauer) filter
Bessel filter
Gaussian filter
Optimum "L" (Legendre) filter

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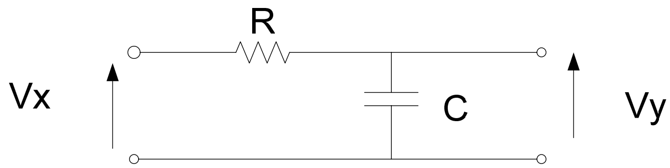
Filter Example:



$$H(s) = \frac{Vx}{Vy}$$

$$H(s) := \frac{1}{1 + R \cdot C \cdot s} \qquad h(t) = \frac{1}{R \cdot C} \cdot e^{\frac{-t}{R \cdot C}} = \frac{1}{\tau} \cdot e^{\frac{-t}{\tau}}$$

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Filter Example:

Let s=j2πf=jω

$$H(\omega) = \frac{1}{1 + \frac{j \cdot \omega}{\omega_o}} \qquad \omega_o := \frac{1}{R \cdot C}$$

$$H(f) = \frac{1}{1 + \frac{j \cdot f}{f_o}} \qquad f_o := \frac{1}{2\pi \cdot R \cdot C}$$

$$\theta(\omega) = \tan^{-1}(\omega \cdot R \cdot C) = \tan^{-1}\left(\frac{\omega}{\omega_o}\right)$$

$$\theta(f) = \tan^{-1}\left(\frac{f}{f_o}\right)$$

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Filter Example:

The Group delay of the RC low pass is:

$$\tau_d(\omega) = \frac{\omega_o}{\omega^2 + \omega_o^2}$$

$$\tau_d(f) = \frac{1}{2\pi} \cdot \frac{f_o}{f^2 + f_o^2}$$

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Figure 2–15 Characteristics of an *RC* low-pass filter.

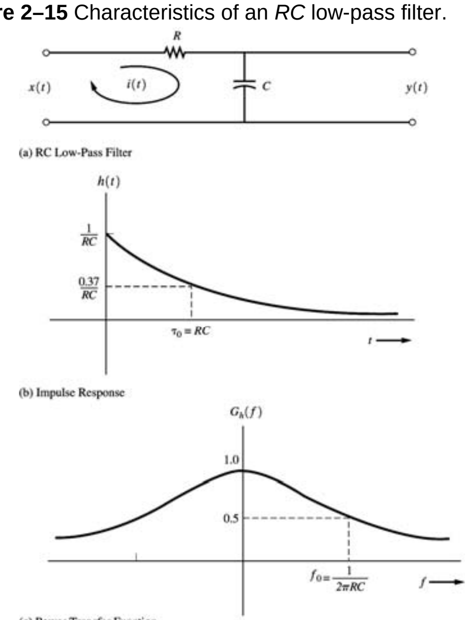

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Figure 2–16 Distortion caused by an *RC* low-pass filter.

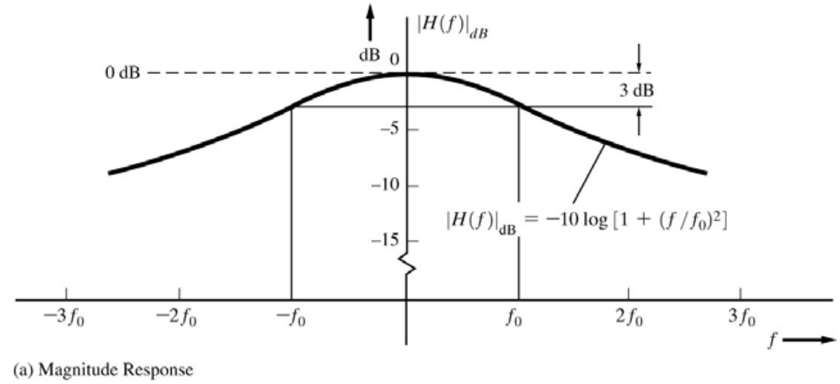


Figure 2–16 Distortion caused by an *RC* low-pass filter.

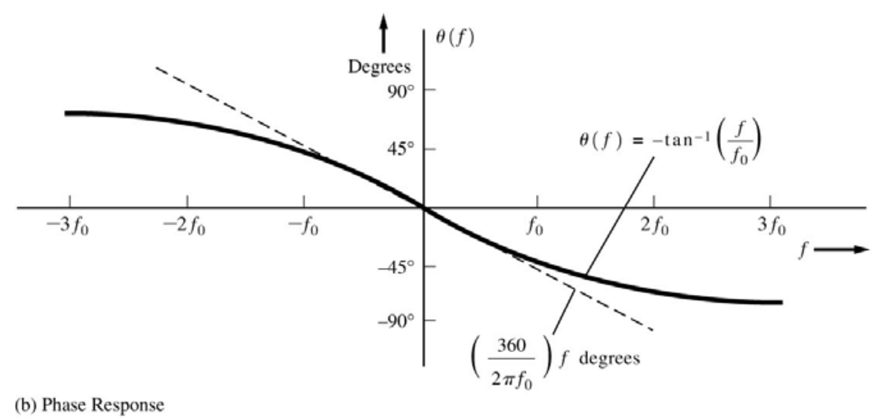
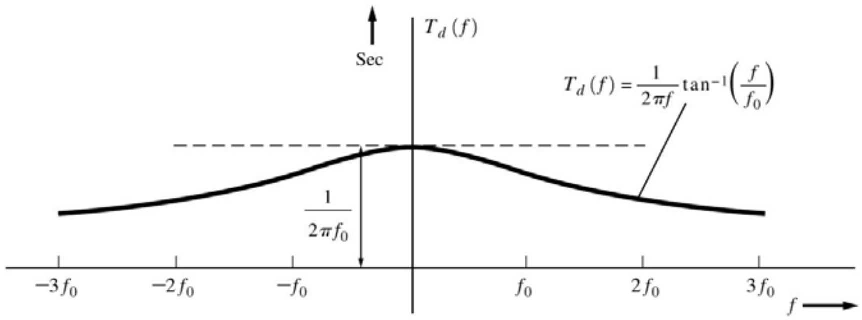


Figure 2–16 Distortion caused by an *RC* low-pass filter.



(c) Time Delay

A LPF Distortion Problem:

- Assume we want the amplitude Linearity <2%
- and the group delay variation (linearity <5%
- Find the usable bandwidth of the 1st order Butterworth filter if the 3dB bandwidth is 1 MHz

A LPF Distortion Problem:

Constraints:

$$\frac{|H(0)| - |H(f_a)|}{|H(0)|} = \varepsilon_a \le 0.02$$ 2% Voltage amplitude error

$$\frac{\tau_d(0) - \tau_d(f_\phi)}{\tau_d(0)} = \varepsilon_\phi \le 0.05$$ 5% delay variation

$f_o := 10^6$ $\quad \tau := \frac{1}{2\pi \cdot f_o}$ $\quad H(f) := \frac{1}{1 + j \cdot \frac{f}{f_o}}$ $\quad \tau_d(f) := \frac{1}{2\pi} \cdot \frac{f_o}{f^2 + f_o{}^2}$

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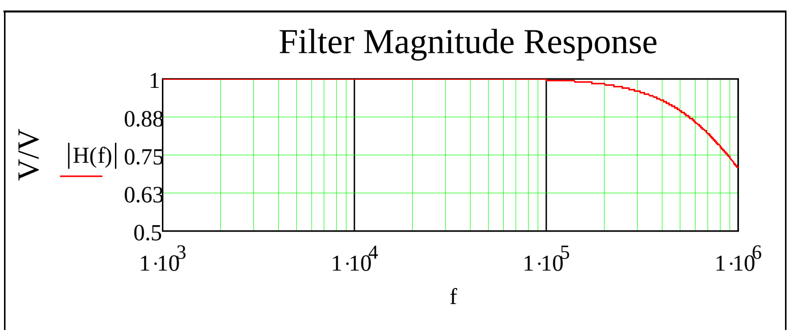
A LPF Distortion Problem:

So the amplitude error will limit the usable bandwidth to 203 KHz



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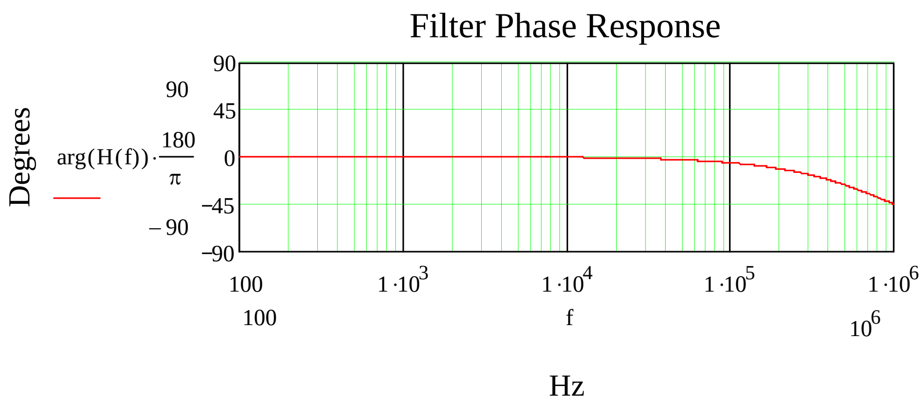
A LPF Distortion Problem:



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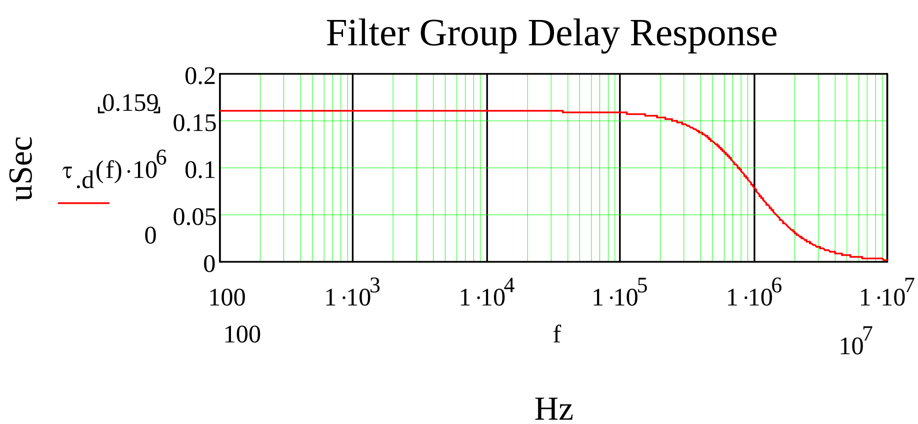
A LPF Distortion Problem:



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A LPF Distortion Problem:

Amplitude Error:

$$\varepsilon_a(f_a) := 1 - \frac{|H(f_a)|}{|H(0)|}$$

$$0.02 = 1 - \frac{1}{\sqrt{1 + \left(\frac{f_a}{f_o}\right)^2}}$$

$$f_a := f_o\sqrt{\left(\frac{1}{0.98}\right)^2 - 1}$$

$$f_a = 2.031 \times 10^5 \ \text{Hz}$$

Phase Error:

t_o

$$\varepsilon_\phi(f_p) := 1 - \frac{\tau_d(f_p)}{\tau_d(0)}$$

$$0.95 = \frac{f_o^{\,2}}{f_p^{\,2} + f_o^{\,2}}$$

$$f_p := \sqrt{\frac{f_o^{\,2}}{0.95} - f_o^{\,2}}$$

$$f_p = 2.294 \times 10^5 \ \text{Hz}$$

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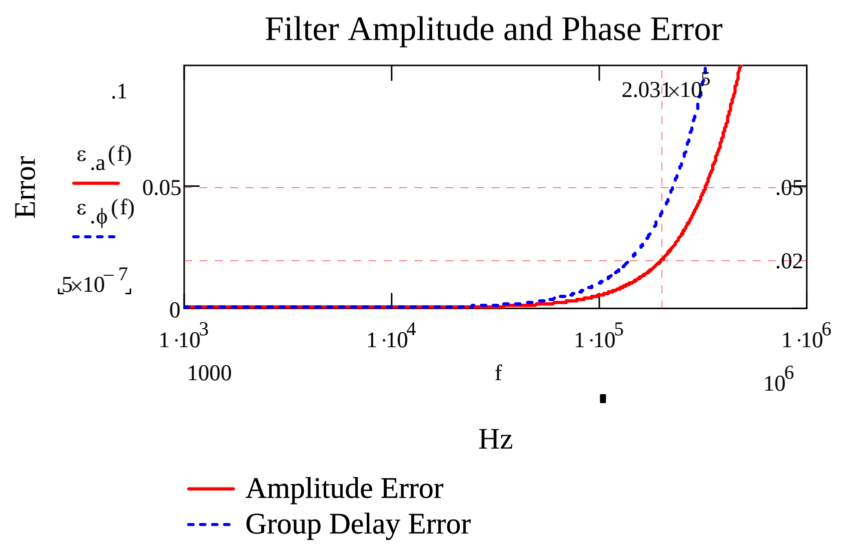
A LPF Distortion Problem: stop 2-3



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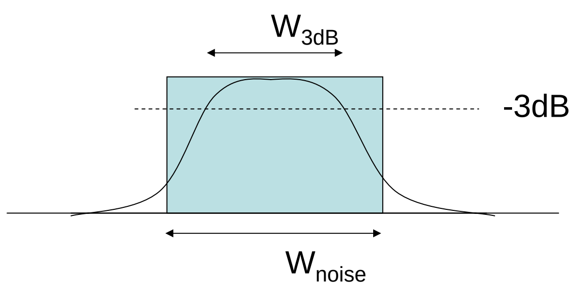
Filter Noise Equivalent Bandwidth

We often equate the -3dB bandwidth of a real
Filter to the bandwidth of an ideal filter that would
Pass the same noise power.



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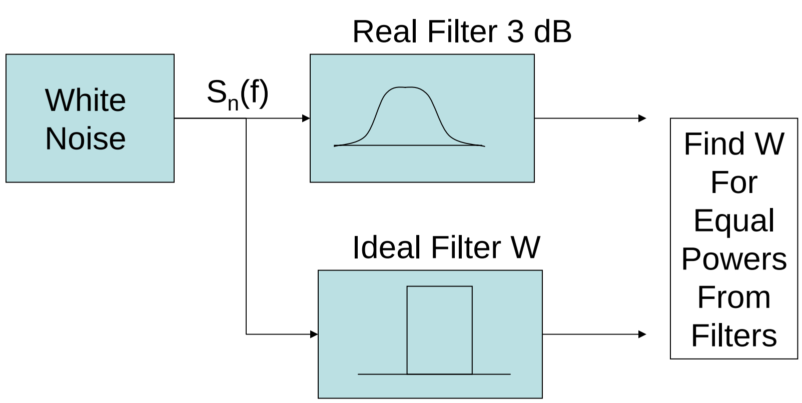
Filter Noise Equivalent Bandwidth



$S_n(f)$ is a White Noise Power Density N_o Watts/Hz

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Filter Noise Equivalent Bandwidth

$$P = \int_{-\infty}^{\infty} S_n(f) \cdot (|H(f)|)^2 \, df = N_o \cdot \int_{0}^{\infty} (|H(f)|)^2 \, df$$

$$P = \int_{-\infty}^{\infty} S_n(f) \cdot \Pi\left(\frac{f}{2W_{eq}}\right)^2 df = \int_{-W_{eq}}^{W_{eq}} \frac{N_o}{2} \, df = N_o \cdot W_{eq}$$

$$W_{eq} = \int_{0}^{\infty} (|H(f)|)^2 \, df$$

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Butterworth lowpass filters

$$(|H(f)|)^2 = \frac{1}{1 + \left(\frac{f}{f_o}\right)^{2 \cdot n}}$$

- Where n is the filter order (number of poles)
- Note that n=1 is the RC lowpass

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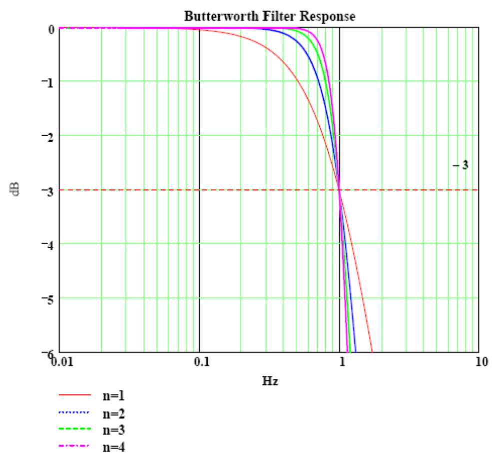
Butterworth lowpass filters

$$H(f,n) := \sqrt{\frac{1}{1+(f)^{2\cdot n}}} \qquad H_{dB}(f,n) := 10\cdot\log\left[\frac{1}{1+(f)^{2\cdot n}}\right]$$



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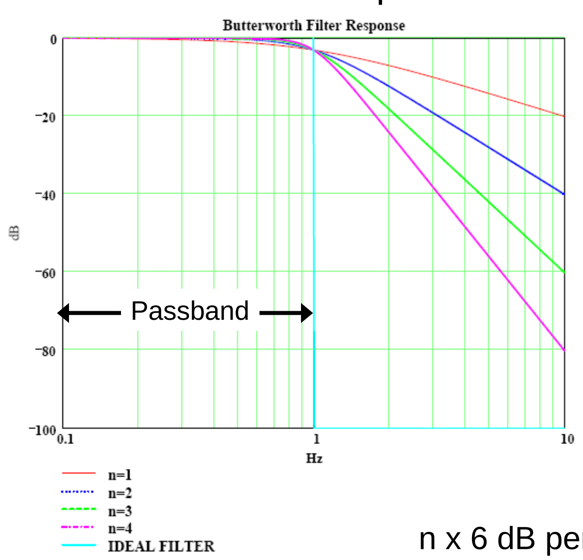
Butterworth lowpass filters



n x 6 dB per octive
n x 20 dB per decade

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Butterworth lowpass filters

EQUIVELENT NOISE BANDWITH FOR BUTTERWORTH FILTERS

$$B = \frac{Noise\_Power\_From\_Real\_Filter\_With\_3db\_Bandwidth\_of\_1}{Noise\_Power\_From\_Ideal\_Filter\_With\_Bandwidth\_of\_1}$$

$n := 1 .. 6$ filter orders from 1 to 6

$$B_n := \frac{\displaystyle\int_0^{\infty} \frac{1}{1+(f)^{2\cdot n}}\,df}{\displaystyle\int_0^{\infty} \Pi(f,1)^2\,df}$$

note that:

$$\int_0^{\infty} \frac{1}{1+(f)^2}\,df = \frac{1}{2}\cdot\pi$$

$n =$	$B_n =$
1	1.571
2	1.111
3	1.047
4	1.026
5	1.017
6	1.012

$$\frac{\pi}{2} = 1.571$$

First order filter with a 3dB bandwidth of 1 passes 57% more noise power when compared with an ideal filter with a bandwidth of 1. A 3rd order filter only passes 4.7% more noise power when compared with an ideal filter.

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