$$L_2 = \frac{1-m^2}{m} L_K = 0.00552 \text{ H} \qquad (1\text{-}7)$$

$$C_2 = mC_K = 0.0460\ \mu\text{F} \qquad (1\text{-}8)$$

(*c*) Compute *m*-derived end sections using $m = 0.6$.

$$L_1 = mL_K = 0.0115 \text{ H} \qquad (1\text{-}6)$$

$$L_2 = \frac{1-m^2}{m} L_K = 0.0204 \text{ H} \qquad (1\text{-}7)$$

$$C_2 = mC_K = 0.0319\ \mu\text{F} \qquad (1\text{-}8)$$

The resulting circuit is shown in figure 1-6*a*. By combining elements, the filter of figure 1-6*b* is obtained.

Constant-*k* and *m*-derived image-parameter design techniques are generally limited. Filters designed by these methods are inefficient and their response characteristics are difficult to predict. Because of these limitations, the subject will not be carried beyond this point.

1.2 MODERN NETWORK THEORY

A generalized filter is shown in figure 1-7. The filter block may consist of inductors, capacitors, resistors, and possibly active elements such as operational amplifiers and transistors. The terminations shown are a voltage source E_s, a source resistance R_s, and a load resistor R_L.

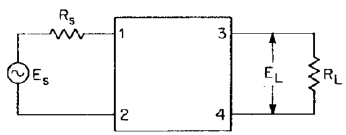

Fig. 1-7 Generalized filter.

The circuit equations for the network of figure 1-7 can be written by using circuit-analysis techniques. Modern network theory solves these equations to determine the network values for optimum performance in some respect.

The Pole-Zero Concept

The frequency response of the generalized filter can be expressed as a ratio of two polynomials in s where $s = j\omega$ ($j = \sqrt{-1}$ and ω, the frequency in radians per second, is $2\pi f$) and is referred to as a transfer function. This can be stated mathematically as

$$T(s) = \frac{E_L}{E_s} = \frac{N(s)}{D(s)} \qquad (1\text{-}9)$$

The roots of the denominator polynomial $D(s)$ are called poles and the roots of the numerator polynomial $N(s)$ are referred to as zeros.

Deriving a network's transfer function could become quite tedious and is beyond the scope of this book. The following discussion explores the evaluation and representation of a relatively simple transfer function.

Analysis of the low-pass filter of figure 1-8*a* results in the following tran function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \qquad (1$$

Let us now evaluate this expression at different frequencies after substitu $j\omega$ for s. The result will be expressed as the absolute magnitude of $T(j\omega)$ the relative attenuation in decibels with respect to the response at DC.

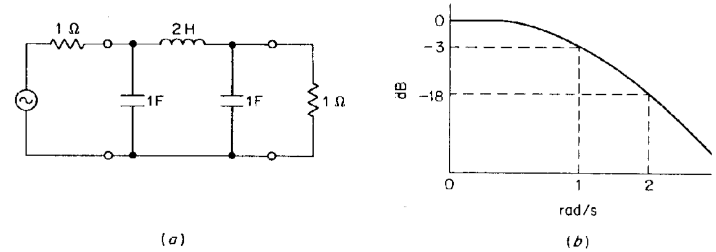

Fig. 1-8 All-pole $N = 3$ low-pass filter: (*a*) filter circuit; (*b*) frequency response.

$$T(j\omega) = \frac{1}{1 - 2\omega^2 + j(2\omega - \omega^3)} \qquad (1\text{-}$$

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0	1	0 dB
1	0.707	−3 dB
2	0.124	−18 dB
3	0.0370	−29 dB
4	0.0156	−36 dB

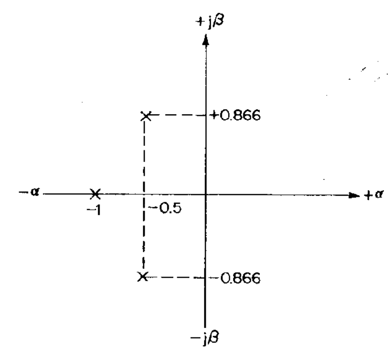

Fig. 1-9 Complex-frequency plane representation of equation (1-10).

The frequency-response curve is plotted in figure 1-8*b*.

Analysis of equation (1-10) indicates that the denominator of the transfer function has three roots or poles and the numerator has none. The filter is therefore called an all-pole type. Since the denominator is a third-order polynomial, the filter is also said to have an $n = 3$ complexity. The denominator poles are $s = -1$, $s = -0.500 + j0.866$, and $s = -0.500 - j0.866$.

These complex numbers can be represented as symbols on a complex-number plane. The abscissa is α, the real component of the root, and the ordinate is β, the imaginary part. Each pole is represented as the symbol X, and a zero is represented as 0. Figure 1-9 illustrates the complex-number plane representation for the roots of equation (1-10).

There are certain mathematical restrictions on the location of poles and zeros in order for the filter to be realizable. They must occur in pairs which are conjugates of each other, except for real-axis poles and zeros, which may occur singly. Poles must also be restricted to the left half plane (i.e., the real coordinate of the pole must be negative). Zeros may occur in either plane.

Synthesis of Filters from Polynomials

Modern network theory has produced families of standard transfer functions that provide optimum filter performance in some desired respect. Synthesis is the process of deriving circuit component values from these transfer functions. Chapter 12 contains extensive tables of transfer functions and their associated component values so that design by synthesis is not required. However, in order to gain some understanding as to how these values have been determined, we will now discuss a few methods of filter synthesis.

Synthesis by Expansion of Driving-Point Impedance The input impedance to the generalized filter of figure 1-7 is the impedance seen looking into terminals 1 and 2 with terminals 3 and 4 terminated, and is referred to as the driving-point impedance or Z_{11} of the network. If an expression for Z_{11} could be determined from the given transfer function, this expression could then be expanded to define the filter.

A family of transfer functions describing the flattest possible shape and a monotonically increasing attenuation in the stopband is the Butterworth low-pass response. These all-pole transfer functions have denominator polynomial roots which fall on a circle having a radius of unity from the origin of the $j\omega$ axis. The attenuation for this family is 3 dB at 1 rad/s.

The transfer function of equation (1-10) satisfies this criterion. It is evident from figure 1-9 that if a circle were drawn having a radius of 1, with the origin as the center, it would intersect the real root and both complex roots.

If R_s in the generalized filter of figure 1-7 is set to 1 Ω, a driving-point impedance expression can be derived in terms of the Butterworth transfer function as

$$Z_{11} = \frac{D(s) - s^n}{D(s) + s^n} \tag{1-12}$$

where $D(s)$ is the denominator polynomial of the transfer function and n is the order of the polynomial.

After $D(s)$ is substituted into equation (1-12), Z_{11} is expanded using the continuous-fraction expansion. This expansion involves successive division and inversion of a ratio of two polynomials. The final form contains a sequence of terms

each alternately representing a capacitor and an inductor and finally the resistive termination. This procedure is demonstrated by the following example.

Example 1-3

REQUIRED: Low-pass LC filter having a Butterworth $n = 3$ response.

RESULT: (*a*) Use the Butterworth transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-10}$$

(*b*) Substitute $D(s) = s^3 + 2s^2 + 2s + 1$ and $s^n = s^3$ into equation (1-12), which results in

$$Z_{11} = \frac{2s^2 + 2s + 1}{2s^3 + 2s^2 + 2s + 1} \tag{1-12}$$

(*c*) Express Z_{11} so that the denominator is a ratio of the higher-order to the lower-order polynomial:

$$Z_{11} = \frac{1}{\dfrac{2s^3 + 2s^2 + 2s + 1}{2s^2 + 2s + 1}}$$

(*d*) Dividing the denominator and inverting the remainder results in

$$Z_{11} = \frac{1}{s + \dfrac{1}{\dfrac{2s^2 + 2s + 1}{s + 1}}}$$

(*e*) After further division and inversion, we get as our final expression

$$Z_{11} = \frac{1}{s + \dfrac{1}{2s + \dfrac{1}{s + 1}}} \tag{1-13}$$

The circuit configuration of figure 1-10 is called a ladder network, since it consists of alternating series and shunt branches. The input impedance can be expressed as the following continued fraction:

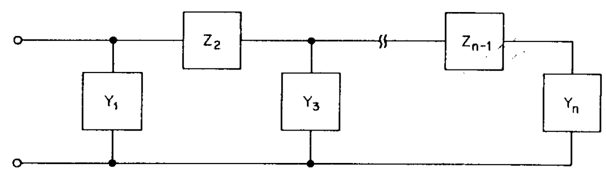

Fig. 1-10 General ladder network.

$$Z_{11} = \frac{1}{Y_1 + \dfrac{1}{Z_2 + \dfrac{1}{Y_3 + \cdots \dfrac{1}{Z_{n-1} + \dfrac{1}{Y_n}}}}} \tag{1-14}$$

where $Y = sC$ and $Z = sL$ for the low-pass all-pole ladder except for a resistive termination where $Y_n = sC + 1/R_L$.

Figure 1-11 can then be derived from equations (1-13) and (1-14) by inspection. This can be proved by reversing the process of expanding Z_{11}. By alternately adding admittances and impedances while working toward the input, Z_{11} is verified as being equal to equation (1-13).

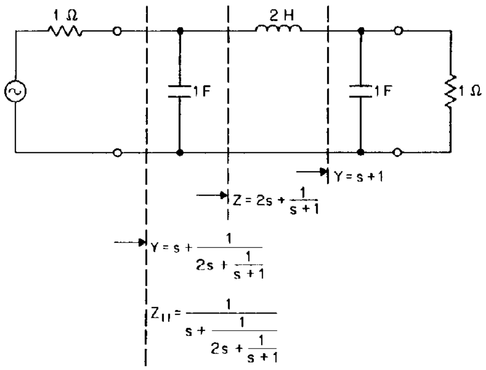

Fig. 1-11 Low-pass filter for equation (1-13).

Synthesis for Unequal Terminations If the source resistor is set equal to 1 Ω and the load resistor is desired to be infinite (unterminated), the impedance looking into terminals 1 and 2 of the generalized filter of figure 1-7 can be expressed as

$$Z_{11} = \frac{D(s\ \text{even})}{D(s\ \text{odd})} \tag{1-15}$$

$D(s$ even) contains all the even-power s terms of the denominator polynomial and $D(s$ odd) consists of all the odd-power s terms of any realizable all-pole low-pass transfer function. Z_{11} is expanded into a continued fraction as in example 1-3 to define the circuit.

Example 1-4

REQUIRED: Low-pass filter having a Butterworth $n = 3$ response with a source resistance of 1 Ω and an infinite termination.

RESULT: *(a)* Use the Butterworth transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-10}$$

(b) Substitute $D(s\ \text{even}) = 2s^2 + 1$ and $D(s\ \text{odd}) = s^3 + 2s$ into equation (1-15):

$$Z_{11} = \frac{2s^2 + 1}{s^3 + 2s} \tag{1-15}$$

(c) Express Z_{11} so that the denominator is a ratio of the higher- to the lower-order polynomial:

$$Z_{11} = \frac{1}{\dfrac{s^3 + 2s}{2s^2 + 1}}$$

(d) Dividing the denominator and inverting the remainder resu

$$Z_{11} = \frac{1}{0.5s + \dfrac{1}{\dfrac{2s^2 + 1}{1.5s}}}$$

(e) Dividing and further inverting results in the final continued tion

$$Z_{11} = \frac{1}{0.5s + \dfrac{1}{1.333s + \dfrac{1}{1.5s}}} \quad ($$

The circuit is shown in figure 1-12.

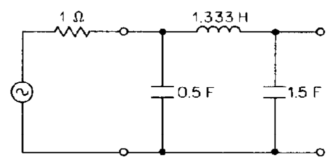

Fig. 1-12 Low-pass filter of example 1-4.

Synthesis by Equating Coefficients An active three-pole low-pass is shown in figure 1-13. Its transfer function is given by

$$T(s) = \frac{1}{s^3A + s^2B + sC + 1} \quad (1$$

where $A = C_1C_2C_3$ (1

$B = 2C_3(C_1 + C_2)$ (1

and $C = C_2 + 3C_3$ (1

If a Butterworth transfer function is desired, we can set equation (1-17) e to equation (1-10).

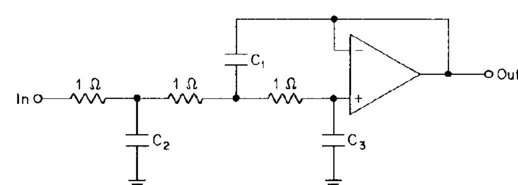

Fig. 1-13 General $N = 3$ active low-pass filter.

$$T(s)=\frac{1}{s^3A+s^2B+sC+1}=\frac{1}{s^3+2s^2+2s+1} \qquad (1\text{-}21)$$

By equating coefficients we obtain

$$A=1$$
$$B=2$$
$$C=2$$

Substituting these coefficients in equations (1-18) through (1-20) and solving for C_1, C_2, and C_3 results in the circuit of figure 1-14.

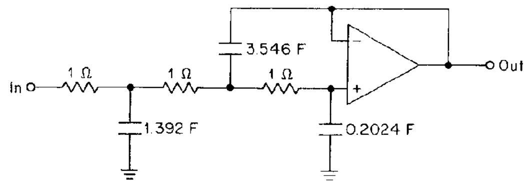

Fig. 1-14 Butterworth $N=3$ active low-pass filter.

Synthesis of filters directly from polynomials offers an elegant solution to filter design. However, it also may involve laborious computations to determine circuit element values. Design methods have been greatly simplified by the curves, tables, and step-by-step procedures provided in this handbook; so design by synthesis can be left to the advanced specialist.

Active versus Passive Filters

The *LC* filters of figures 1-11 and 1-12 and the active filter of figure 1-14 all satisfy an $n=3$ Butterworth low-pass transfer function. The filter designer is frequently faced with the sometimes difficult decision of choosing whether to use an active or *LC* design. A number of factors must be considered. Some of the limitations and considerations for each filter type will now be discussed.

Frequency Limitations At subaudio frequencies, *LC* filter designs require high values of inductance and capacitance along with their associated bulk. Active filters are more practical because they can be designed at higher impedance levels so that capacitor magnitudes are reduced.

Above 50 kHz, most commercial-grade operational amplifiers have insufficient open-loop gain for the average active filter requirement. However, amplifiers are available with extended bandwidth at increased cost so that active filters at frequencies up to 500 kHz are possible. *LC* filters, on the other hand, are practical at frequencies up to a few hundred megahertz. Beyond this range, filters become impractical to build in lumped form, and so distributed parameter techniques are used.

Size Considerations Active filters are generally smaller than their *LC* counterparts, since inductors are not required. Further reduction in size is possible with microelectronic technology. By using deposited *RC* networks and monolithic operational amplifier chips or with hybrid technology, active filters can be reduced to microscopic proportions.

Economics and Ease of Manufacture *LC* filters generally cost more t active filters because they use inductors. High-quality coils require efficient n netic cores. Sometimes, special coil-winding methods are needed. These fac lead to the increased cost of *LC* filters.

Active filters have the distinct advantage that they can be easily asseml using standard off-the-shelf components. *LC* filters require coil-winding coil-assembly skills.

Ease of Adjustment In critical *LC* filters, tuned circuits require adjustn to specific resonances. Capacitors cannot be made variable unless they are b a few hundred picofarads. Inductors, however, can easily be adjusted, s most coil structures provide a means for tuning such as an adjustment slu

Many active filter circuits are not easily adjustable. They may contain sections where two or more resistors in each section have to be varied in o to control resonances. These circuits have been avoided. The active filter de techniques presented in this handbook include convenient methods for adjus resonances where required such as for narrow-band bandpass filters.

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2

Selecting the Response Characteristic

2.1 FREQUENCY-RESPONSE NORMALIZATION

Several parameters are used to characterize a filter's performance. The commonly specified requirement is frequency response. When given a frequu response specification, the engineer must select a filter design that meets t requirements. This is accomplished by transforming the required respon a normalized low-pass specification having a cutoff of 1 rad/s. This norma response is compared with curves of normalized low-pass filters which also a 1 rad/s cutoff. After a satisfactory low-pass filter is determined from the cu the tabulated normalized element values of the chosen filter are transfoı or denormalized to the final design.

Modern network theory has provided us with many different shapes of aı tude versus frequency which have been analytically derived by placing va restrictions on transfer functions. The major categories of these low-pas sponses are:

Butterworth
Chebyshev
Linear phase
Transitional
Synchronously tuned
Elliptic-function

With the exception of the elliptic-function family, these responses are all no ized to a 3-dB cutoff of 1 rad/s.

Frequency and Impedance Scaling

The basis for normalization of filters is the fact that a given filter's resp can be scaled (shifted) to a different frequency range by dividing the reɛ elements by a frequency-scaling factor (FSF). The FSF is the ratio of a refeı frequency of the desired response to the corresponding reference frequ of the given filter. Usually 3-dB points are selected as reference freque of low-pass and high-pass filters and the center frequency is chosen as the reference for bandpass filters. The FSF can be expressed as

$$\text{FSF} = \frac{\text{desired reference frequency}}{\text{existing reference frequency}} \qquad (2\text{-}1)$$

The FSF must be a dimensionless number; so both the numerator and denominator of equation (2-1) must be expressed in the same units, usually radians per second. The following example demonstrates computation of the FSF and frequency scaling of filters.

Example 2-1

REQUIRED: Low-pass filter either *LC* or active with $n = 3$ Butterworth transfer function having a 3-dB cutoff at 1000 Hz.

RESULT: Figure 2-1 illustrates the *LC* and active $n = 3$ Butterworth low-pass filters discussed in chapter 1 and their response.

(*a*) Compute FSF.

$$\text{FSF} = \frac{2\pi 1000 \text{ rad/s}}{1 \text{ rad/s}} = 6280 \qquad (2\text{-}1)$$

(*b*) Dividing all the reactive elements by the FSF results in the filters of figure 2-2 *a* and *b* and the response of figure 2-2 *c*.

Note that all points on the frequency axis of the normalized response have been multiplied by the FSF. Also since the normalized filter has its cutoff at 1 rad/s, the FSF can be directly expressed by $2\pi f_c$, where f_c is the desired low-pass cutoff frequency in hertz.

Frequency scaling a filter has the effect of multiplying all points on the frequency axis of the response curve by the FSF. Therefore, a normalized response curve can be directly used to predict the attenuation of the denormalized filter.

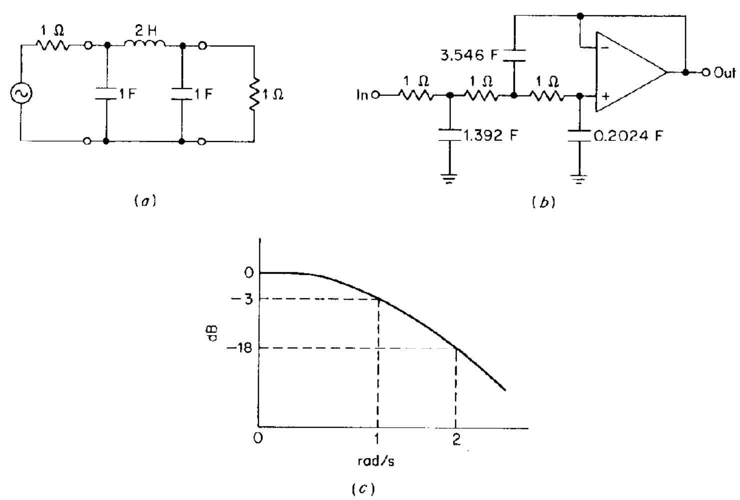


Fig. 2-1 $N = 3$ Butterworth low-pass filter: (*a*) *LC* filter; (*b*) active filter; (*c*) frequency response.

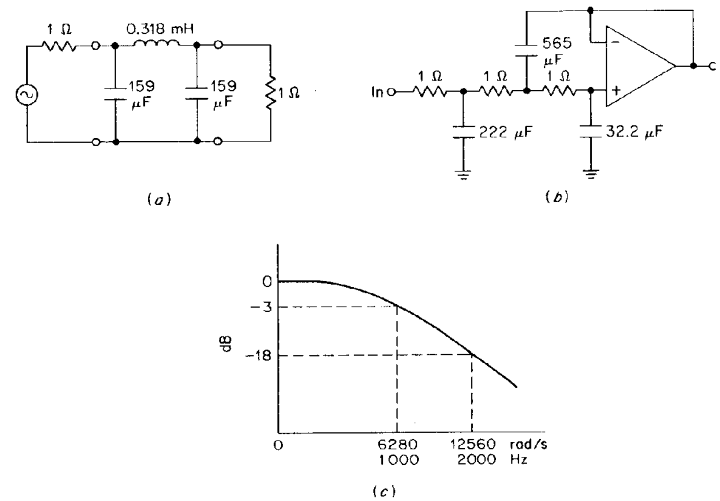


Fig. 2-2 Denormalized low-pass filter of example 2-1: (*a*) *LC* filter; (*b*) active filter; frequency response.

When the filters of figure 2-1 were denormalized to those of figure 2-2, transfer function changed as well. The denormalized transfer function beca

$$T(s) = \frac{1}{4.03 \times 10^{-12}s^3 + 5.08 \times 10^{-9}s^2 + 3.18 \times 10^{-4}s + 1} \qquad (\text{2}$$

The denominator has the roots: $s = -6280$, $s = -3140 + j5438$, and $-3140 - j5438$.

These roots can be obtained directly from the normalized roots by multiply the normalized root coordinates by the FSF. Frequency scaling a filter a scales the poles and zeros (if any) by the same factor.

The component values of the filters in figure 2-2 are not very practical. T capacitor values are much too large and the $1 - \Omega$ resistor values are not v desirable. This situation can be resolved by impedance scaling. Any linear ac or passive network maintains its transfer function if all resistor and induc values are multiplied by an impedance-scaling factor *Z* and all capacitors divided by the same factor *Z*. This occurs because the *Z*'s cancel in the tran

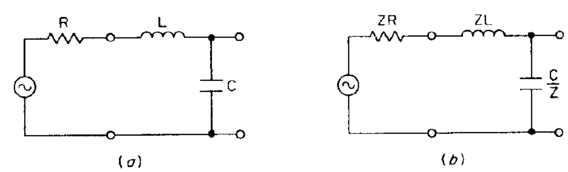


Fig. 2-3 Two-pole low-pass *LC* filter: (*a*) basic filter; (*b*) impedance-scaled filter.

function. To prove this, let us investigate the transfer function of the simple two-pole low-pass filter of figure 2-3*a*, which is

$$T(s)=\frac{1}{s^2LC+sCR+1} \tag{2-3}$$

Impedance scaling can be mathematically expressed as

$$R'=ZR \tag{2-4}$$

$$L'=ZL \tag{2-5}$$

$$C'=\frac{C}{Z} \tag{2-6}$$

where the primes denote the values after impedance scaling.

If we impedance-scale the filter, we obtain the circuit of figure 2-3*b*. The new transfer function becomes

$$T(s)=\frac{1}{s^2ZL\frac{C}{Z}+s\frac{C}{Z}ZR+1} \tag{2-7}$$

Clearly, the Z's cancel; so both transfer functions are equivalent.

We can now use impedance scaling to make the values in the filters of figure 2-2 more practical. If we use impedance scaling with a Z of 1000, we obtain the filters of figure 2-4. The values are certainly more suitable.

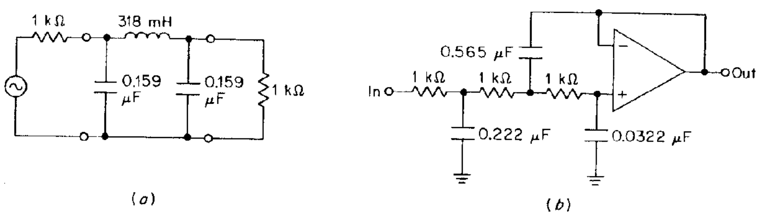


Fig. 2-4 Impedance-scaled filters of example 2-1: *(a)* LC filter; *(b)* active filter.

Frequency and impedance scaling are normally combined into one step rather than performed sequentially. The denormalized values are then given by

$$R'=R\times Z \tag{2-8}$$

$$L'=\frac{L\times Z}{\text{FSF}} \tag{2-9}$$

$$C'=\frac{C}{\text{FSF}\times Z} \tag{2-10}$$

where the primed values are both frequency- and impedance-scaled.

Low-Pass Normalization

In order to use normalized low-pass filter curves and tables, a given low-pass filter requirement must first be converted into a normalized requirement. The

curves can now be entered to find a satisfactory normalized filter which is tl scaled to the desired cutoff.

The first step in selecting a normalized design is to convert the requirem into a steepness factor A_s, which can be defined as

$$A_s=\frac{f_s}{f_c} \qquad \text{(2-}$$

where f_s is the frequency having the minimum required stopband attenuat and f_c is the limiting frequency or cutoff of the passband, usually the 3- point. The normalized curves are compared with A_s, and a design is selec that meets or exceeds the requirement. The design is then frequency-sca so that the selected passband limit of the normalized design occurs at f_c.

If the required passband limit f_c is defined as the 3-dB cutoff, the steepn factor A_s can be directly looked up in radians per second on the freque axis of the normalized curves.

Suppose that we required a low-pass filter that has a 3-dB point at 100 and more than 30 dB attenuation at 400 Hz. A normalized low-pass filter tl has its 3-dB point at 1 rad/s and over 30 dB attenuation at 4 rad/s wo meet the requirement if the filter were frequency-scaled so that the 3-dB po occurred at 100 Hz. Then there would be over 30 dB attenuation at 400 I or 4 times the cutoff, because a response shape is retained when a filter frequency-scaled.

The following example demonstrates normalizing a simple low-pass requi ment.

Example 2-2

REQUIRED: Normalize the following specification:
Low-pass filter
3 dB at 200 Hz
30 dB minimum at 800 Hz

RESULT: *(a)* Compute A_s.

$$A_s=\frac{f_s}{f_c}=\frac{800\text{ Hz}}{200\text{ Hz}}=4 \qquad \text{(2-11}$$

(b) Normalized requirement:
3 dB at 1 rad/s
30 dB minimum at 4 rad/s

In the event f_c does not correspond to the 3-dB cutoff, A_s can still be comput and a normalized design found that will meet the specifications. This is illustrat in the following example.

Example 2-3

REQUIRED: Normalize the following specification:
Low-pass filter
1 dB at 200 Hz
30 dB minimum at 800 Hz

RESULT: *(a)* Compute A_s.

$$A_s=\frac{f_s}{f_c}=\frac{800\text{ Hz}}{200\text{ Hz}}=4 \qquad \text{(2-11)}$$

(b) Normalized requirement:
1 dB at K rad/s
30 dB minimum at $4K$ rad/s
(where K is arbitrary)

function. To prove this, let us investigate the transfer function of the simple two-pole low-pass filter of figure 2-3*a*, which is

$$T(s)=\frac{1}{s^2LC+sCR+1} \tag{2-3}$$

Impedance scaling can be mathematically expressed as

$$R' = ZR \tag{2-4}$$

$$L' = ZL \tag{2-5}$$

$$C' = \frac{C}{Z} \tag{2-6}$$

where the primes denote the values after impedance scaling.

If we impedance-scale the filter, we obtain the circuit of figure 2-3*b*. The new transfer function becomes

$$T(s)=\frac{1}{s^2ZL\frac{C}{Z}+s\frac{C}{Z}ZR+1} \tag{2-7}$$

Clearly, the Z's cancel; so both transfer functions are equivalent.

We can now use impedance scaling to make the values in the filters of figure 2-2 more practical. If we use impedance scaling with a Z of 1000, we obtain the filters of figure 2-4. The values are certainly more suitable.

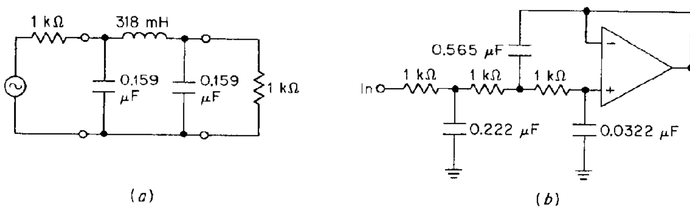


Fig. 2-4 Impedance-scaled filters of example 2-1: (*a*) *LC* filter; (*b*) active filter.

Frequency and impedance scaling are normally combined into one step rather than performed sequentially. The denormalized values are then given by

$$R' = R \times Z \tag{2-8}$$

$$L' = \frac{L \times Z}{\text{FSF}} \tag{2-9}$$

$$C' = \frac{C}{\text{FSF} \times Z} \tag{2-10}$$

where the primed values are both frequency- and impedance-scaled.

Low-Pass Normalization

In order to use normalized low-pass filter curves and tables, a given low-pass filter requirement must first be converted into a normalized requirement. The curves can now be entered to find a satisfactory normalized filter which is then scaled to the desired cutoff.

The first step in selecting a normalized design is to convert the requirement into a steepness factor A_s, which can be defined as

$$A_s = \frac{f_s}{f_c} \tag{2-11}$$

where f_s is the frequency having the minimum required stopband attenuation and f_c is the limiting frequency or cutoff of the passband, usually the 3-dB point. The normalized curves are compared with A_s, and a design is selected that meets or exceeds the requirement. The design is then frequency-scaled so that the selected passband limit of the normalized design occurs at f_c.

If the required passband limit f_c is defined as the 3-dB cutoff, the steepness factor A_s can be directly looked up in radians per second on the frequency axis of the normalized curves.

Suppose that we required a low-pass filter that has a 3-dB point at 100 Hz and more than 30 dB attenuation at 400 Hz. A normalized low-pass filter that has its 3-dB point at 1 rad/s and over 30 dB attenuation at 4 rad/s would meet the requirement if the filter were frequency-scaled so that the 3-dB point occurred at 100 Hz. Then there would be over 30 dB attenuation at 400 Hz, or 4 times the cutoff, because a response shape is retained when a filter is frequency-scaled.

The following example demonstrates normalizing a simple low-pass requirement.

Example 2-2

REQUIRED: Normalize the following specification:
Low-pass filter
3 dB at 200 Hz
30 dB minimum at 800 Hz

RESULT: (*a*) Compute A_s.

$$A_s = \frac{f_s}{f_c} = \frac{800\text{ Hz}}{200\text{ Hz}} = 4 \tag{2-11}$$

(*b*) Normalized requirement:
3 dB at 1 rad/s
30 dB minimum at 4 rad/s

In the event f_c does not correspond to the 3-dB cutoff, A_s can still be computed and a normalized design found that will meet the specifications. This is illustrated in the following example.

Example 2-3

REQUIRED: Normalize the following specification:
Low-pass filter
1 dB at 200 Hz
30 dB minimum at 800 Hz

RESULT: (*a*) Compute A_s.

$$A_s = \frac{f_s}{f_c} = \frac{800\text{ Hz}}{200\text{ Hz}} = 4 \tag{2-11}$$

(*b*) Normalized requirement:
1 dB at K rad/s
30 dB minimum at $4K$ rad/s
(where K is arbitrary)

A possible solution to example 2-3 would be a normalized filter which has a 1-dB point at 0.8 rad/s and over 30 dB attenuation at 3.2 rad/s. The fundamental requirement is that the normalized filter makes the transition between the passband and stopband limits within a frequency ratio A_s.

High-Pass Normalization

A normalized $n = 3$ low-pass Butterworth transfer function was given in section 1.2 as

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \qquad (1\text{-}10)$$

and the results of evaluating this transfer function at various frequencies were:

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0	1	0 dB
1	0.707	−3 dB
2	0.124	−18 dB
3	0.0370	−29 dB
4	0.0156	−36 dB

Let us now perform a high-pass transformation by substituting $1/s$ for s in equation (1-10). After some algebraic manipulations the resulting transfer function becomes

$$T(s) = \frac{s^3}{s^3 + 2s^2 + 2s + 1} \qquad (2\text{-}12)$$

If we evaluate this expression at specific frequencies, we can generate the following table:

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0.25	0.0156	−36 dB
0.333	0.0370	−29 dB
0.500	0.124	−18 dB
1	0.707	−3 dB
∞	1	0 dB

The response is clearly that of a high-pass filter. It is also apparent that the low-pass attenuation values now occur at high-pass frequencies that are exactly the reciprocals of the corresponding low-pass frequencies. A high-pass transformation of a normalized low-pass filter transposes the low-pass attenuation values to reciprocal frequencies and retains the 3-dB cutoff at 1 rad/s. This relationship is evident in figure 2-5, where both filter responses are compared.

The normalized low-pass curves could be interpreted as normalized high-pass curves by reading the attenuation as indicated and taking the reciprocals of the frequencies. However, it is much easier to convert a high-pass specification into a normalized low-pass requirement and use the curves directly.

To normalize a high-pass filter specification calculate A_s, which in the (of high-pass filters is given by

$$A_s = \frac{f_c}{f_s} \qquad (2\cdot$$

Since the A_s for high-pass filters is defined as the reciprocal of the A_s for l pass filters, equation (2-13) can be directly interpreted as a low-pass requirem

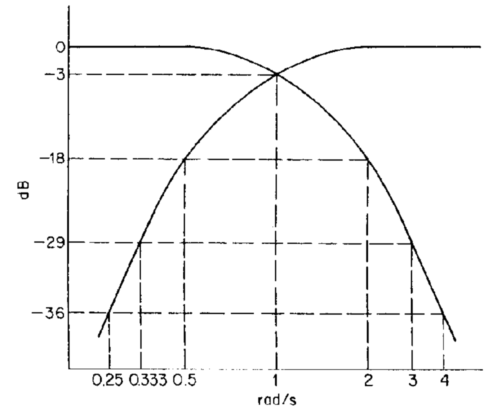


Fig. 2-5 Normalized low-pass high-pass relationship.

A normalized low-pass filter can then be selected from the curves. A high-| transformation is performed on the corresponding low-pass filter, and the re: ing high-pass filter is scaled to the desired cutoff frequency.

The following example shows the normalization of a high-pass filter reqı ment.

Example 2-4

REQUIRED: Normalize the following requirement:
High-pass filter
3 dB at 200 Hz
30 dB minimum at 50 Hz

RESULT: (a) Compute A_s.

$$A_s = \frac{f_c}{f_s} = \frac{200 \text{ Hz}}{50 \text{ Hz}} = 4 \qquad (2\text{-}1$$

(b) Normalized equivalent low-pass requirement:
3 dB at 1 rad/s
30 dB minimum at 4 rad/s

Bandpass Normalization

Bandpass filters fall into two categories, narrow-band and wide-band. If ratio of the upper cutoff frequency to the lower cutoff frequency is over 2 octave), the filter is considered a wide-band type.

Wide-Band Bandpass Filters Wide-band filter specifications can be separated into individual low-pass and high-pass requirements which are treated independently. The resulting low-pass and high-pass filters are then cascaded to meet the composite response.

Example 2-5

REQUIRED: Normalize the following specification:
Bandpass filter
3 dB at 500 and 1000 Hz
40 dB minimum at 200 and 2000 Hz

RESULT: *(a)* Determine the ratio of upper cutoff to lower cutoff.

$$\frac{1000 \text{ Hz}}{500 \text{ Hz}} = 2$$

wide-band type

(b) Separate requirement into individual specifications.

High-pass filter:	Low-pass filter:
3 dB at 500 Hz	3 dB at 1000 Hz
40 dB minimum at 200 Hz	40 dB minimum at 2000 Hz
$A_s = 2.5$ (2-13)	$A_s = 2.0$ (2-11)

(c) Normalized high-pass and low-pass filters are now selected, scaled to the required cutoff frequencies, and cascaded to meet the composite requirements. Figure 2-6 shows the resulting circuit and response.

Narrow-Band Bandpass Filters Narrow-band bandpass filters have a ratio of upper cutoff frequency to lower cutoff frequency of approximately 2 or less

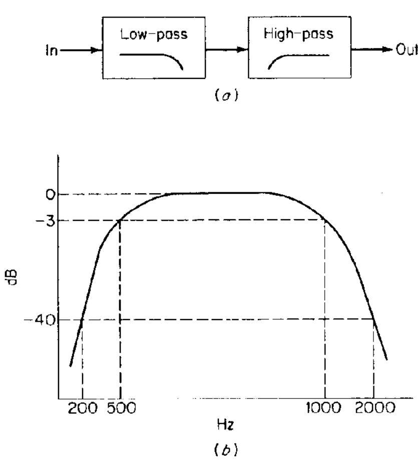


Fig. 2-6 Results of example 2-5: *(a)* cascade of low-pass and high-pass filters; *(b)* frequency response.

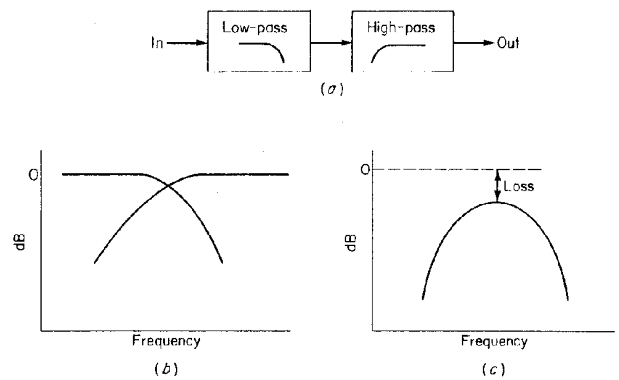


Fig. 2-7 Limitation of wide-band approach for narrow-band filters: *(a)* cascade of low-pass and high-pass filters; *(b)* composite response; *(c)* algebraic sum of attenuation.

and cannot be designed as separate low-pass and high-pass filters. The m reason for this is evident from figure 2-7. As the ratio of upper cutoff to lc cutoff decreases, the loss at center frequency will increase, and it may bec prohibitive for ratios near unity.

If we substitute $s + 1/s$ for s in a low-pass transfer function, a bandpass f results. The center frequency occurs at 1 rad/s, and the frequency respc of the low-pass filter is directly transformed into the bandwidth of the band filter at points of equivalent attenuation. In other words, the attenuation b width ratios remain unchanged. This is shown in figure 2-8, which shows relationship between a low-pass filter and its transformed bandpass equival Each pole and zero of the low-pass filter is transformed into a *pair* of p and zeros in the bandpass filter.

In order to design a bandpass filter, the following sequence of steps is invol

1. Convert the given bandpass filter requirement into a normalized low-specification.
2. Select a satisfactory low-pass filter from the normalized frequency-respc curves.
3. Transform the normalized low-pass parameters into the required band filter.

The response shape of a bandpass filter is shown in figure 2-9 along some basic terminology. The center frequency is defined as

$$f_0 = \sqrt{f_L f_u} \qquad \text{(2}$$

where f_L is the lower passband limit and f_u is the upper passband limit, usu the 3-dB attenuation frequencies. For the more general case

$$f_0 = \sqrt{f_1 f_2} \qquad \text{(2}$$

where f_1 and f_2 are any two frequencies having equal attenuation. These relat ships imply geometric symmetry; that is, the entire curve below f_0 is the mi image of the curve above f_0 when plotted on a *logarithmic* frequency axis.

An important parameter of bandpass filters is the filter selectivity factor or Q, which is defined as

$$Q = \frac{f_0}{\mathrm{BW}} \tag{2-16}$$

where BW is the passband bandwidth or $f_u - f_L$.

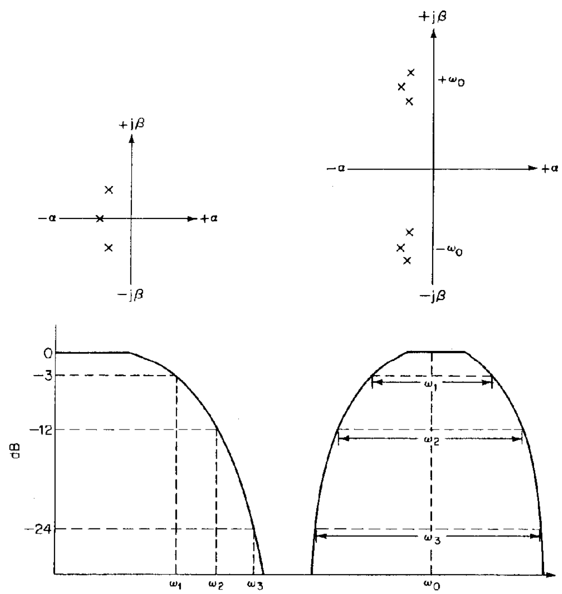


Fig. 2-8 Low-pass to bandpass transformation.

As the filter Q increases, the response shape near the passband approaches the arithmetically symmetrical condition, i.e., mirror-image symmetry near the center frequency, when plotted using a *linear* frequency axis. For Q's of 10 or more the center frequency can be redefined as the arithmetic mean of the passband limits; so we can replace equation (2-14) with

$$f_0 = \frac{f_L + f_u}{2} \tag{2-17}$$

In order to utilize the normalized low-pass filter frequency-response curves, a given narrow-band bandpass filter specification must be transformed into a normalized low-pass requirement. This is accomplished by first manipulating

the specification to make it geometrically symmetrical. At equivalent attenu points, corresponding frequencies above and below f_0 must satisfy

$$f_1 f_2 = f_0^2 \quad (\text{:}$$

which is an alternate form of equation (2-15) for geometric symmetry. given specification is modified by calculating the corresponding opposite ge

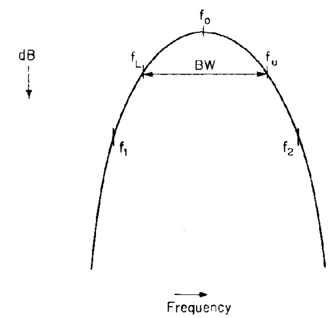


Fig. 2-9 General bandpass filter response shape.

tric frequency for each stopband frequency specified. Each pair of stopl frequencies will result in two new frequency pairs. The pair having the l separation is retained, since it represents the more severe requirement.

A bandpass filter steepness factor can now be defined as

$$A_s = \frac{\text{stopband bandwidth}}{\text{passband bandwidth}} \quad (\text{:}$$

This steepness factor is used to select a normalized low-pass filter from frequency-response curves that makes the passband to stopband transition w a frequency ratio of A_s.

The following example illustrates the normalization of a bandpass filte quirement.

Example 2-6

REQUIRED: Normalize the following bandpass filter requirement:
Bandpass filter
Center frequency of 100 Hz
3 dB at ±15 Hz (85 Hz, 115 Hz)
40 dB at ±30 Hz (70 Hz, 130 Hz)

RESULT: (*a*) First compute center frequency f_0.

$$f_0 = \sqrt{f_L f_u} = \sqrt{85 \times 115} = 98.9 \text{ Hz} \quad (2\text{-}1$$

(*b*) Compute two geometrically related stopband frequency pair each pair of stopband frequencies given.

Let $f_1 = 70$ Hz.

$$f_2 = \frac{f_0^2}{f_1} = \frac{(98.9)^2}{70} = 139.7 \text{ Hz} \qquad (2\text{-}18)$$

Let $f_2 = 130$ Hz.

$$f_1 = \frac{f_0^2}{f_2} = \frac{(98.9)^2}{130} = 75.2 \text{ Hz} \qquad (2\text{-}18)$$

The two pairs are

$f_1 = 70$ Hz, $f_2 = 139.7$ Hz ($f_2 - f_1 = 69.7$ Hz)

and $f_1 = 75.2$ Hz, $f_2 = 130$ Hz ($f_2 - f_1 = 54.8$ Hz)

Retain the second frequency pair, since it has the lesser separation. Figure 2-10 compares the specified filter requirement and the geometrically symmetrical equivalent.

(c) Calculate A_s.

$$A_s = \frac{\text{stopband bandwidth}}{\text{passband bandwidth}} = \frac{54.8 \text{ Hz}}{30 \text{ Hz}} = 1.83 \qquad (2\text{-}19)$$

(d) A normalized low-pass filter can now be selected from the normalized curves. Since the passband limit is the 3-dB point, the normalized filter is required to have over 40 dB of rejection at 1.83 rad/s or 1.83 times the 1-rad/s cutoff.

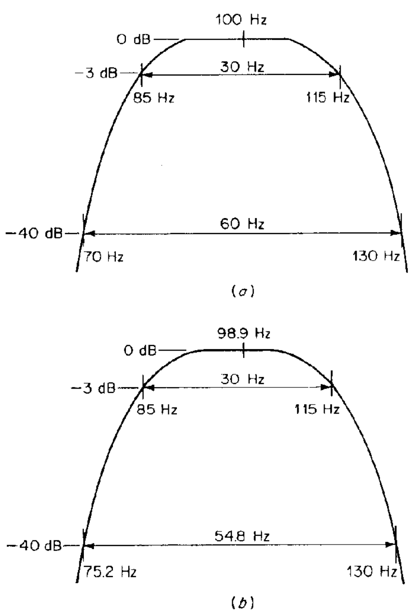


Fig. 2-10 Frequency-response requirements of example 2-6: *(a)* given filter requirement; *(b)* geometrically symmetrical requirement.

Bandpass filter requirements are not always specified in an arithmetically metrical manner as in the previous example. Multiple stopband attenuatic quirements may also exist. The design engineer is still faced with the problem of converting the given parameters into geometrically symmetrical acteristics so that a steepness factor or factors can be determined. The follc example demonstrates conversion of a specification somewhat more compli than the previous example.

Example 2-7

REQUIRED: Normalize the following bandpass filter specification:
Bandpass filter
1 dB passband limits of 12 and 14 kHz
20 dB minimum at 6 kHz
30 dB minimum at 4 kHz
40 dB minimum at 56 kHz

RESULT: *(a)* First compute the center frequency.

$f_L = 12$ kHz $\quad f_u = 14$ kHz

$f_0 = 12.96$ kHz (2-

(b) Compute the corresponding geometric frequency for each band frequency given, using equation (2-18).

$$f_1 f_2 = f_0^2 \qquad (2\text{-}$$

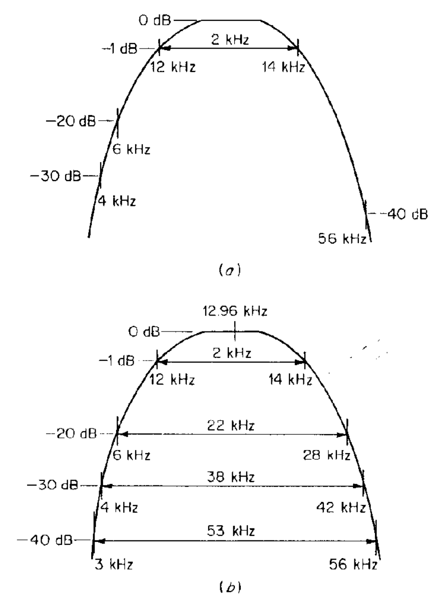
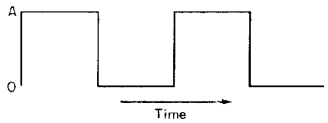


Fig. 2-11 Given and transformed response of example 2-7: *(a)* given requirement; *(b)* geometrically symmetrical response.

It is well known that a square wave can be represented by a Fourier series of odd harmonic components as indicated in figure 2-18. Since the amplitude of each harmonic is reduced as the harmonic order increases, only the first



$$A(t) = A\left(\frac{1}{2} + \frac{2}{\pi}\cos\omega_1\tau - \frac{2}{3\pi}\cos 3\omega_1\tau + \frac{2}{5\pi}\cos 5\omega_1\tau + \ldots\right)$$

Fig. 2-18 Frequency analysis of a square wave.

few harmonics are of significance. If a square wave is applied to a filter, the fundamental and its significant harmonics must have the proper relative amplitude relationship at the filter's output in order to retain the square waveshape. In addition, these components must not be displaced in time with respect to each other. Let us now consider the effect of a low-pass filter's phase shift on a square wave.

If we assume that a low-pass filter has a linear phase shift between 0° at DC and n times $-45°$ at the cutoff, we can express the phase shift in the passband as

$$\phi = -\frac{45 n f_x}{f_c} \tag{2-21}$$

where f_x is any frequency in the passband and f_c is the 3-dB cutoff frequency.

A phase-shifted sine wave appears displaced in time from the input waveform. This displacement is called "phase delay" and can be computed by determining the time interval represented by the phase shift, using the fact that a full period contains 360°. Phase delay can then be computed by

$$T_{pd} = \frac{\phi}{360}\frac{1}{f_x} \tag{2-22}$$

or, as an alternate form,

$$T_{pd} = -\frac{\beta}{\omega} \tag{2-23}$$

where β is the phase shift in radians (1 rad = $360/2\pi$ or 57.3°) and ω is the input frequency expressed in radians per second ($\omega = 2\pi f_x$).

Example 2-10

REQUIRED: Compute the phase delay of the fundamental and the third, fifth, seventh, and ninth harmonics of a 1-kHz square wave applied to an $n = 3$ Butterworth low-pass filter having a 3-dB cutoff of 10 kHz. Assume a linear phase shift with frequency in the passband.

RESULT: Using formulas (2-21) and (2-22), the following table can be computed:

Frequency	ϕ	T_{pd}
1 kHz	−13.5°	37.5 μs
3 kHz	−40.5°	37.5 μs
5 kHz	−67.5°	37.5 μs
7 kHz	−94.5°	37.5 μs
9 kHz	−121.5°	37.5 μs

The phase delays of the fundamental and each of the significant harmon in example 2-10 are identical. The output waveform would then appear nea equivalent to the input except for a delay of 37.5 μs. If the phase shift is r linear with frequency, the ratio ϕ/f_x in equation (2-22) is not constant; so ea significant component of the input square wave would undergo a different del This displacement in time of the spectral components, with respect to ea other, introduces a distortion of the output waveform. Figure 2-19 shows so typical effects of nonlinear phase shift upon a square wave. Most filters ha nonlinear phase versus frequency characteristics; so some waveform distorti will usually occur for complex input signals.

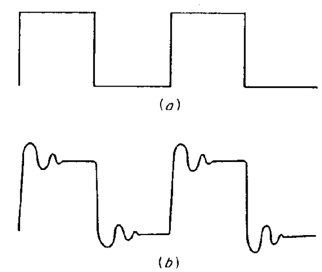


Fig. 2-19 Effect of nonlinear phase: *(a)* ideal square wave; *(b)* distorted square wave.

Not all complex waveforms have harmonically related spectral componen An amplitude-modulated signal, for example, consists of a carrier and two sic bands, each sideband separated from the carrier by the modulating frequen If a filter's phase characteristic is linear with frequency and intersects zero pha shift at zero frequency (DC), both the carrier and the two sidebands will ha the same delay in passing through the filter; so the output will be a delay replica of the input. If these conditions are not satisfied, the carrier and bo sidebands will be delayed by different amounts. The carrier delay will be accordance with the equation for phase delay

$$T_{pd} = -\frac{\beta}{\omega} \tag{2-23}$$

(The terms carrier delay and phase delay are used interchangeably.)

A new definition is required for the delay of the sidebands. This delay commonly called "group delay" and is defined as the derivative of phase vers frequency, which can be expressed as

$$T_{gd} = -\frac{d\beta}{d\omega} \tag{2-24}$$

Linear phase shift results in constant group delay, since the derivative of a linear function is a constant. Figure 2-20 illustrates a low-pass filter phase shift which is nonlinear in the vicinity of a carrier ω_c and the two sidebands $\omega_c - \omega_m$ and $\omega_c + \omega_m$. The phase delay at ω_c is the negative slope of a line drawn

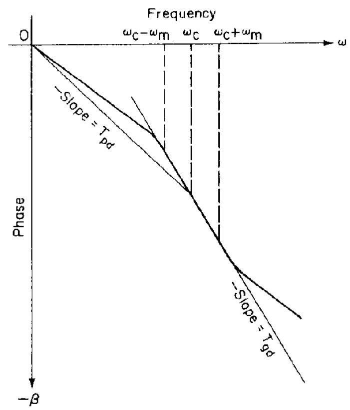


Fig. 2-20 Nonlinear phase shift of a low-pass filter.

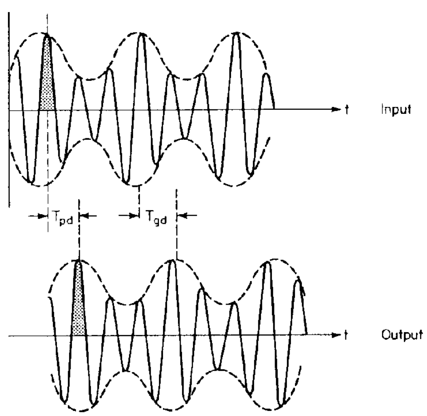


Fig. 2-21 Effect of nonlinear phase on AM signal.

from the origin to the phase shift corresponding to ω_c which is in agreem with equation (2-23). The group delay at ω_c is shown as the negative slope a line which is tangent to the phase response at ω_c. This can be mathematica expressed as

$$T_{gd} = -\frac{d\beta}{d\omega}\bigg|_{\omega=\omega_c}$$

If the two sidebands are restricted to a region surrounding ω_c having a const group delay, the envelope of the modulated signal will be delayed by T_{gd}. Fig 2-21 compares the input and output waveforms of an amplitude-modulated sig applied to the filter depicted by figure 2-20. Note that the carrier is delay by the phase delay while the envelope is delayed by the group delay. For t reason group delay is sometimes called "envelope delay."

If the group delay is not constant over the bandwidth of the modulated sign waveform distortion will occur. Narrow-bandwidth signals are more likely encounter constant group delay than signals having a wider spectrum. It common practice to use group-delay variation as a criterion to evaluate ph nonlinearity and subsequent waveform distortion. The absolute magnitude the nominal delay is usually of little consequence.

Step Response of Networks

If we were to define a hypothetical ideal low-pass filter, it would have the respon shown in figure 2-22. The amplitude response is unity from DC to the cut frequency ω_c and zero beyond the cutoff. The phase shift is a linearly increasi function in the passband, where n is the order of the ideal filter. The gro delay is constant in the passband and zero in the stopband. If a unity amplitu step were applied to this ideal filter at $t = 0$, the output would be in accordan

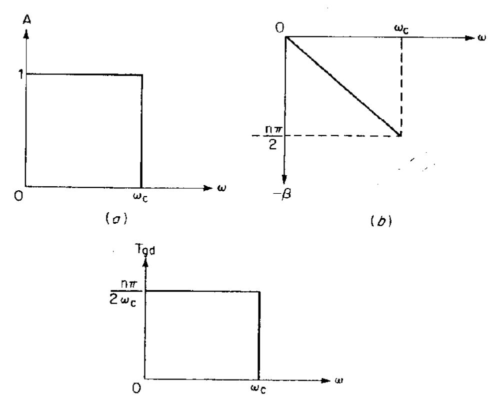
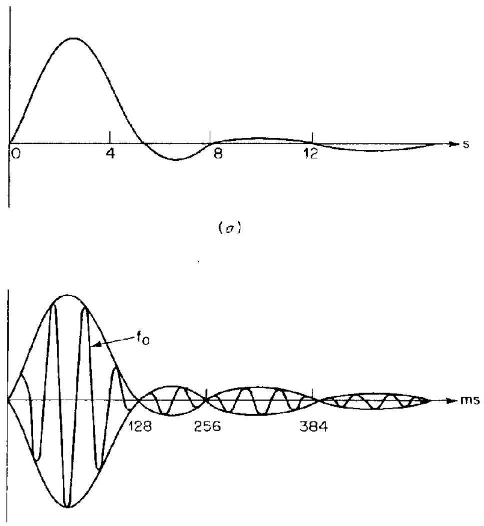


Fig. 2-22 Ideal low-pass filter: *(a)* frequency response; *(b)* phase shift; *(c)* group delay.

sponse, divide the normalized low-pass rise time by πBW, where BW is 10 Hz. The resulting rise time is approximately 120 ms, which well exceeds the burst duration. Also, 10 cycles of center frequency occur during the burst interval; so the impulse response can be used to approximate the output envelope. To denormalize the impulse response, multiply the amplitude axis by πBW and divide the time axis by the same factor. The results are shown in figure 2-32.



(b)

Fig. 2-32 Results of example 2-17: *(a)* normalized low-pass impulse response; *(b)* impulse response of bandpass filter.

Effective Use of the Group-Delay, Step-Response, and Impulse-Response Curves Many signals consist of complex forms of modulation rather than pulses or steps; so the transient response curves cannot be directly used to estimate the amount of distortion introduced by the filters. However, the curves are useful as a figure of merit, since networks having desirable step- or impulse-response behavior introduce minimal distortion to most forms of modulation.

Examination of the step- and impulse-response curves in conjunction with group delay indicates that a necessary condition for good pulse transmission is a flat group delay. A gradual transition from the passband to the stopband is also required for low transient distortion but is highly undesirable from a frequency-attenuation point of view.

In order to obtain a rapid pulse rise time the higher-frequency spectral components should not be delayed with respect to the lower frequencies. The curves indicate that low-pass filters which do have sharply increasing delay at higher frequencies have an impulse response which comes to a peak at a later time.

When a low-pass filter is transformed to a high-pass, a band-reject, or a wide-band bandpass filter the transient properties are not preserved. Lindquist and

Zverev (see references) provide computational methods for the calculatioı these responses.

2.3 BUTTERWORTH MAXIMALLY FLAT AMPLITUDE

The Butterworth approximation to an ideal low-pass filter is based on the sumption that a flat response at zero frequency is more important than response at other frequencies. The normalized transfer function is an all-ı type having roots which all fall on a unit circle. The attenuation is 3 dB rad/s.

The attenuation of a Butterworth low-pass filter can be expressed by

$$A_{dB} = 10 \log \left[1 + \left(\frac{\omega_x}{\omega_c} \right)^{2n} \right] \qquad (2\text{-}$$

where ω_x/ω_c is the ratio of the given frequency ω_x to the 3-dB cutoff freque ω_c and n is the order of the filter.

For the more general case,

$$A_{dB} = 10 \log (1 + \Omega^{2n}) \qquad (2\text{-}$$

where Ω is defined by the following table:

Filter Type	Ω
Low-pass	ω_x/ω_c
High-pass	ω_c/ω_x
Bandpass	$BW_x/BW_{3\,dB}$
Band-reject	$BW_{3\,dB}/BW_x$

The value Ω is a dimensionless ratio of frequencies or normalized freque $BW_{3\,dB}$ is the 3-dB bandwidth and BW_x is the bandwidth of interest. At h values of Ω the attenuation increases at a rate of $6n$ dB per octave, where octave is defined as a frequency ratio of 2 for the low-pass and high-pass c and a *bandwidth* ratio of 2 for bandpass and band-reject filters.

The pole positions of the normalized filter all lie on a unit circle and be computed by

$$-\sin \frac{(2K-1)\pi}{2n} + j \cos \frac{(2K-1)\pi}{2n}, \qquad K = 1,2, \ldots, n \qquad (2\text{-}$$

and the element values for an LC normalized low-pass filter operating betw equal 1-Ω terminations can be calculated by

$$L_K \text{ or } C_K = 2 \sin \frac{(2K-1)\pi}{2n}, \qquad K = 1,2, \ldots, n \qquad (2\text{-}$$

where $(2K-1)\pi/2n$ is in radians.

Equation (2-31) is exactly equal to twice the real part of the pole positi of equation (2-30) except that the sign is positive.

Example 2-18

REQUIRED: Calculate the frequency response at 1,2, and 4 rad/s, the pole p tions, and the LC element values of a normalized $n = 5$ Butterworth low-ı filter.

RESULT: *(a)* Using equation (2-29) with $n = 5$, the following frequency-response table can be derived:

Ω	Attenuation
1	3 dB
2	30 dB
4	60 dB

(b) The pole positions are computed using equation (2-30) as follows:

K	$-\sin \frac{(2K-1)\pi}{2n}$	$j \cos \frac{(2K-1)\pi}{2n}$
1	−0.309	$+j0.951$
2	−0.809	$+j0.588$
3	−1	
4	−0.809	$-j0.588$
5	−0.309	$-j0.951$

(c) The element values can be computed by equation (2-31) and have the following values:

$L_1 = 0.618$ H		$C_1 = 0.618$ F
$C_2 = 1.618$ F		$L_2 = 1.618$ H
$L_3 = 2$ H	or	$C_3 = 2$ F
$C_4 = 1.618$ F		$L_4 = 1.618$ H
$L_5 = 0.618$ H		$C_5 = 0.618$ F

The results of example 2-18 are shown in figure 2-33.

Chapter 12 provides pole locations and element values for both *LC* and active Butterworth low-pass filters having complexities up to $n = 10$.

The Butterworth approximation results in a class of filters which have moderate attenuation steepness and acceptable transient characteristics. Their element values are more practical and less critical than those of most other filter types. The rounding of the frequency response in the vicinity of cutoff may make these filters undesirable where a sharp cutoff is required, but nevertheless they should be used wherever possible because of their favorable characteristics.

Figures 2-34 through 2-37 indicate the frequency response, group delay, impulse response, and step response for the Butterworth family of low-pass filters normalized to a 3-dB cutoff of 1 rad/s.

2.4 CHEBYSHEV RESPONSE

If the poles of the normalized Butterworth low-pass transfer function were moved to the right by multiplying the real parts of the pole positions by a constant k_c where $k_c < 1$, the poles would now lie on an ellipse instead of a unit circle. The frequency response would ripple evenly and have a 3-dB cutoff of 1 rad/s. As the real part of the poles is decreased by lowering k_c, the ripples will grow in magnitude. The resulting response is called the Chebyshev or equiripple function.

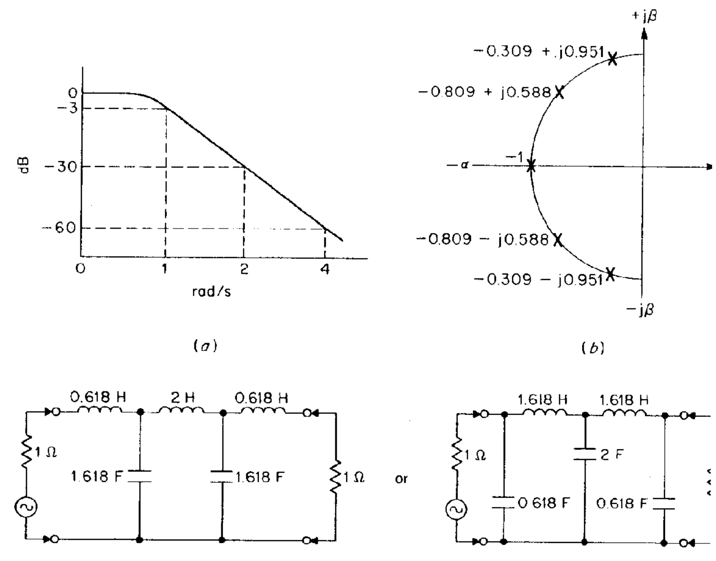


Fig. 2-33 Butterworth low-pass filter of example 2-18: *(a)* frequency response; *(b)* p locations; *(c)* circuit configuration.

The Chebyshev approximation to an ideal filter has a much more rectangu frequency response in the region near cutoff than the Butterworth family filters. This is accomplished at the expense of allowing ripples in the passba

The factor k_c can be computed by

$$k_c = \tanh A \qquad (2\text{-}$$

The parameter A is given by

$$A = \frac{1}{n} \cosh^{-1} \frac{1}{\epsilon} \qquad (2\text{-}$$

where

$$\epsilon = \sqrt{10^{R_{dB}/10} - 1} \qquad (2\text{-}$$

and R_{db} is the ripple in decibels.

Figure 2-38 compares the voltage response of an $n = 3$ Butterworth normaliz low-pass filter and the Chebyshev filter generated by multiplying the real pa of the roots by k_c. Both filters have half-power (3-dB) bandwidths of 1 rad The ripple bandwidth of the Chebyshev filter is $1/\cosh A$.

The attenuation of Chebyshev filters can be expressed as

$$A_{dB} = 10 \log [1 + \epsilon^2 C_n^2(\Omega)] \qquad (2\text{-}$$

where $C_n(\Omega)$ is a Chebyshev polynomial whose magnitude oscillates betwe ± 1 for $\Omega \leq 1$. Table 2-1 lists the Chebyshev polynomials up to order $n =$

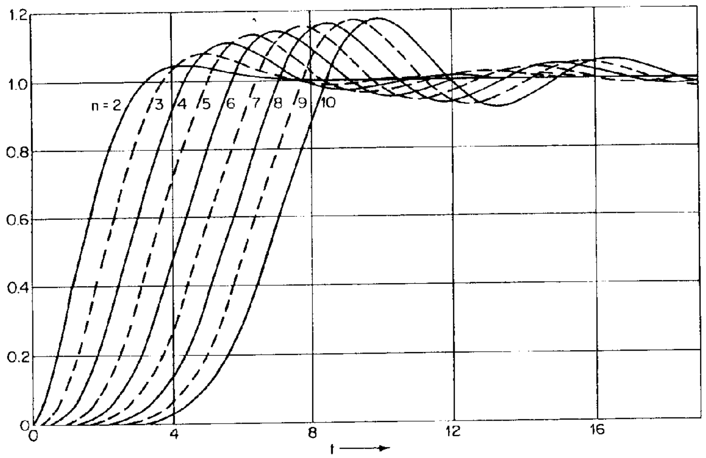


Fig. 2-37 Step response for Butterworth filters. *(From Anatol I. Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 1967. By permission of the publishers.)*

TABLE 2-1 Chebyshev Polynomials

1.	Ω
2.	$2\Omega^2 - 1$
3.	$4\Omega^3 - 3\Omega$
4.	$8\Omega^4 - 8\Omega^2 + 1$
5.	$16\Omega^5 - 20\Omega^3 + 5\Omega$
6.	$32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$
7.	$64\Omega^7 - 112\Omega^5 + 56\Omega^3 - 7\Omega$
8.	$128\Omega^8 - 256\Omega^6 + 160\Omega^4 - 32\Omega^2 + 1$
9.	$256\Omega^9 - 576\Omega^7 + 432\Omega^5 - 120\Omega^3 + 9\Omega$
10.	$512\Omega^{10} - 1280\Omega^8 + 1120\Omega^6 - 400\Omega^4 + 50\Omega^2 - 1$

At $\Omega = 1$, Chebyshev polynomials have a value of unity; so the attenuation defined by equation (2-35) would be equal to the ripple. The 3-dB cutoff is slightly above $\Omega = 1$ and is equal to cosh A. In order to normalize the response equation so that 3 dB of attenuation occurs at $\Omega = 1$, the Ω of equation (2-35) is computed by using the following table:

Filter Type	Ω
Low-pass	(cosh A) ω_x/ω_c
High-pass	(cosh A) ω_c/ω_x
Bandpass	(cosh A) $BW_x/BW_{3\ dB}$
Band-reject	(cosh A) $BW_{3\ dB}/BW_x$

Figure 2-39 compares the ratios of 3-dB bandwidth to ripple bandwidth (cosh A) for Chebyshev low-pass filters ranging from $n = 2$ through $n = 10$.

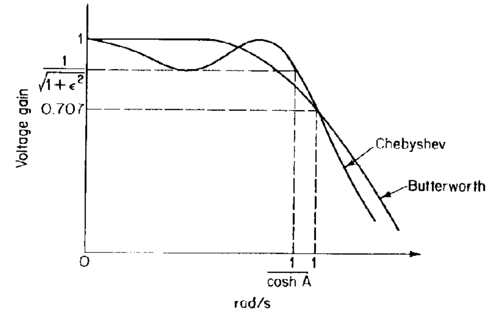


Fig. 2-38 Comparison of Butterworth and Chebyshev low-pass filters.

n	0.001 dB	0.005 dB	0.01 dB	0.05 dB
2	5.7834930	3.9027831	3.3036192	2.2685899
3	2.6427081	2.0740079	1.8771819	1.5120983
4	1.8416695	1.5656920	1.4669048	1.2783955
5	1.5155888	1.3510908	1.2912179	1.1753684
6	1.3495755	1.2397596	1.1994127	1.1207360
7	1.2531352	1.1743735	1.1452685	1.0882424
8	1.1919877	1.1326279	1.1106090	1.0673321
9	1.1507149	1.1043196	1.0870644	1.0530771
10	1.1215143	1.0842257	1.0703312	1.0429210
n	0.10 dB	0.25 dB	0.50 dB	1.00 dB
2	1.9432194	1.5981413	1.3897437	1.2176261
3	1.3889948	1.2528880	1.1674852	1.0948680
4	1.2130992	1.1397678	1.0931019	1.0530019
5	1.1347180	1.0887238	1.0592591	1.0338146
6	1.0929306	1.0613406	1.0410296	1.0234422
7	1.0680005	1.0449460	1.0300900	1.0172051
8	1.0519266	1.0343519	1.0230107	1.0131638
9	1.0409547	1.0271099	1.0181668	1.0103963
10	1.0331307	1.0219402	1.0147066	1.0084182

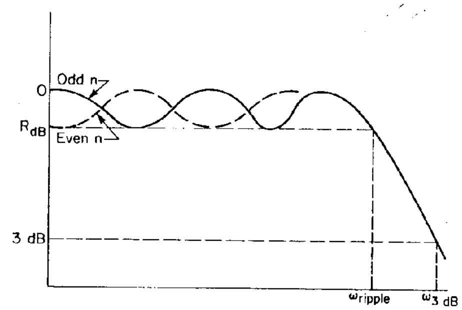


Fig. 2-39 Ratio of 3-dB bandwidth to ripple bandwidth.

Odd-order Chebyshev LC filters have zero relative attenuation at DC. Even-order filters, however, have a loss at DC equal to the passband ripple. As a result the even-order networks must operate between unequal source and load resistances, whereas for odd n's, the source and load may be equal. However, a mathematical transformation can alter even-order networks for operation between equal terminations (see $\theta = T$ in table 12-56). The result is a Chebyshev-like behavior in the passband and a slightly diminished rate of roll-off when compared with a comparable unaltered network.

The element values for an LC normalized low-pass filter operating between equal 1-Ω terminations and having an odd n can be calculated from the following series of relations:

$$G_1 = \frac{2A_1 \cosh A}{Y} \tag{2-36}$$

$$G_k = \frac{4A_{k-1}A_k \cosh^2 A}{B_{k-1}G_{k-1}} \qquad k = 2, 3, 4, \ldots, n \tag{2-37}$$

where

$$Y = \sinh \frac{\beta}{2n} \tag{2-38}$$

$$\beta = \ln\left(\coth \frac{R_{dB}}{17.37}\right) \tag{2-39}$$

$$A_k = \sin \frac{(2k-1)\pi}{2n} \qquad k = 1, 2, 3, \ldots, n \tag{2-40}$$

$$B_k = Y^2 + \sin^2\left(\frac{k\pi}{n}\right) \qquad k = 1, 2, 3, \ldots, n \tag{2-41}$$

Coefficients G_1 through G_n are the element values.

An alternate form of determining LC element values is by synthesis of the driving-point impedance directly from the transfer function. This method includes both odd- and even-order n's.

Example 2-19

REQUIRED: Compute the pole positions, the frequency response at 1, 2, and 4 rad/s, and the element values of a normalized $n = 5$ Chebyshev low-pass filter having a ripple of 0.5 dB.

RESULT: *(a)* To compute the pole positions, first solve for k_c as follows:

$$\epsilon = \sqrt{10^{R_{dB}/10} - 1} = 0.349 \tag{2-34}$$

$$A = \frac{1}{n} \cosh^{-1} \frac{1}{\epsilon} = 0.355 \tag{2-33}$$

$$k_c = \tanh A = 0.340 \tag{2-32}$$

Multiplication of the real parts of the normalized Butterworth poles of example 2-18 by k_c results in the following new pole positions:

$-0.105 \pm j0.951$
$-0.275 \pm j0.588$
-0.34

(b) To calculate the frequency response, substitute a fifth-order Chebyshev polynomial and $\epsilon = 0.349$ into equation (2-35). The following results are obtained:

Ω	A_{dB}
1.0	3 dB
2.0	45 dB
4.0	77 db

(c) The element values are computed as follows:

$A_1 = 0.309$ (2-4
$\beta = 3.55$ (2-3
$Y = 0.363$ (2-3
$G_1 = 1.81$ (2-3
$G_2 = 1.30$ (2-3
$G_3 = 2.69$ (2-3
$G_4 = 1.30$ (2-3
$G_5 = 1.81$ (2-3

Coefficients G_1 through G_5 represent the element values of a malized Chebyshev low-pass filter having a 0.5-dB ripple a 3-dB cutoff of 1 rad/s.

Figure 2-40 shows the results of this example.

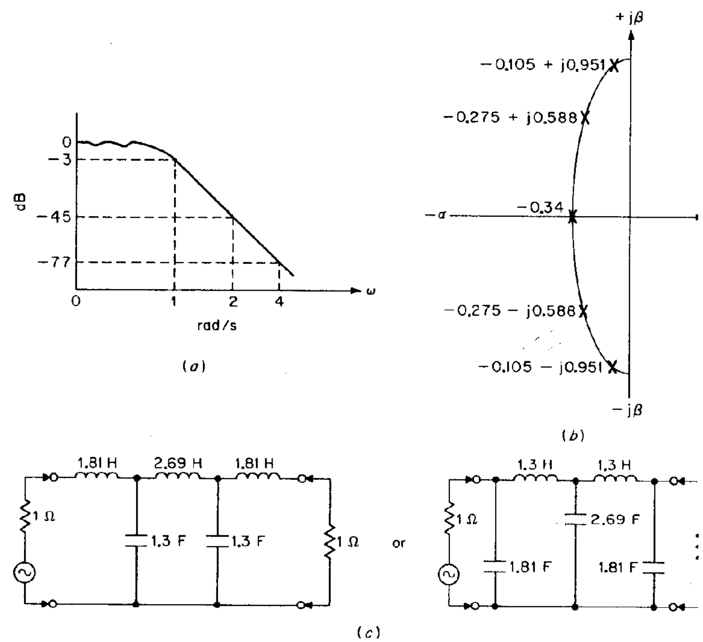


Fig. 2-40 Chebyshev low-pass filter of example 2-19: *(a)* frequency response; *(b)* p locations; *(c)* circuit configuration.

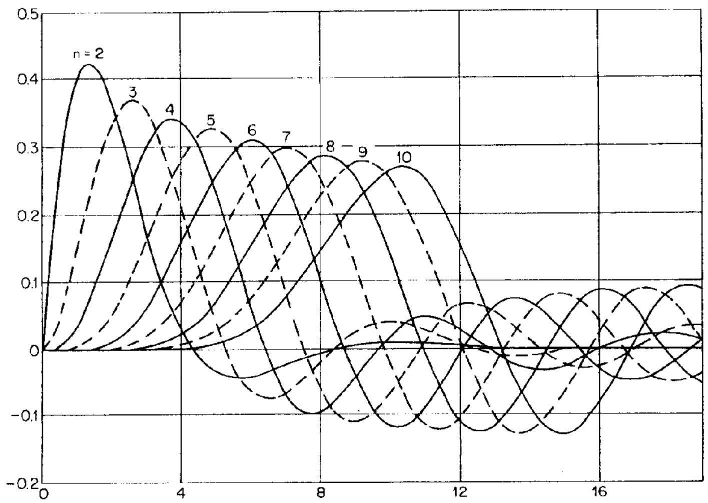


Fig. 2-53 Impulse response for Chebyshev filters with 0.5-dB ripple. *(From Anatol I. Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 1967. By permission of the publishers.)*

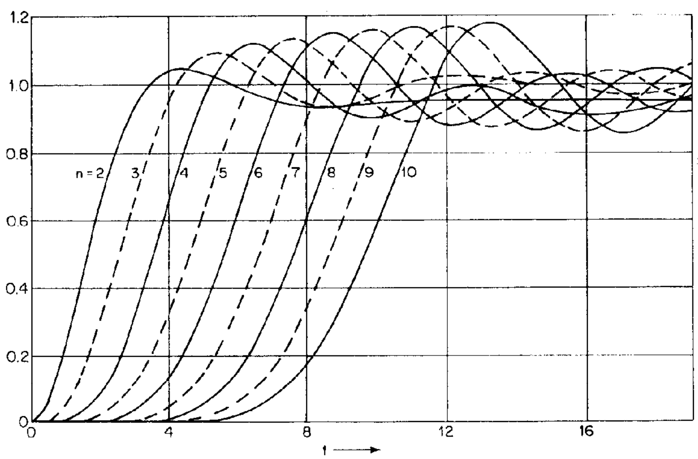


Fig. 2-54 Step response for Chebyshev filters with 0.5-dB ripple. *(From Anatol I. Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 1967. By permission of the publishers.)*

2.5 BESSEL MAXIMALLY FLAT DELAY

Butterworth filters have fairly good amplitude and transient characteristics. Chebyshev family of filters offers increased selectivity but poor transient be ior. Neither approximation to an ideal filter is directed toward obtaining a stant delay in the passband.

The Bessel transfer function has been optimized to obtain a linear pl i.e., a maximally flat delay. The step response has essentially no overshoc ringing and the impulse response lacks oscillatory behavior. However, the quency response is much less selective than in the other filter types.

The low-pass approximation to a constant delay can be expressed as the fol ing general transfer function:

$$T(s)=\frac{1}{\sinh s+\cosh s} \qquad (2$$

If a continued-fraction expansion is used to approximate the hyperbolic fi tions and the expansion is truncated at different lengths, the Bessel famil transfer functions will result.

A crude approximation to the pole locations can be found by locating the poles on a circle and separating their imaginary parts by $2/n$, as show figure 2-55. The vertical spacing between poles is equal, whereas in the Bu worth case the angles were equal.

The relative attenuation of a Bessel low-pass filter can be approximatec

$$A_{dB}=3\left(\frac{\omega_x}{\omega_c}\right)^2 \qquad (2\cdot$$

This expression is reasonably accurate for ω_x/ω_c ranging between 0 an Figures 2-56 through 2-59 indicate that as the order n is increased, the reg of flat delay is extended farther into the stopband. However, the steepnes roll-off in the transition region does not improve significantly. This restr the use of Bessel filters to applications where the transient properties are major consideration.

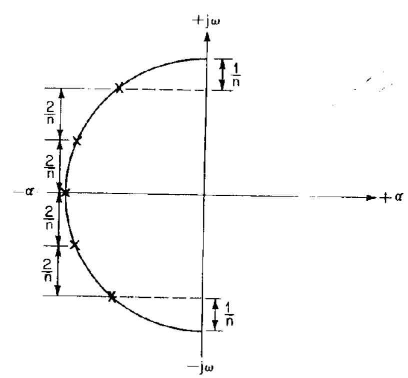


Fig. 2-55 Approximate Bessel pole locations.

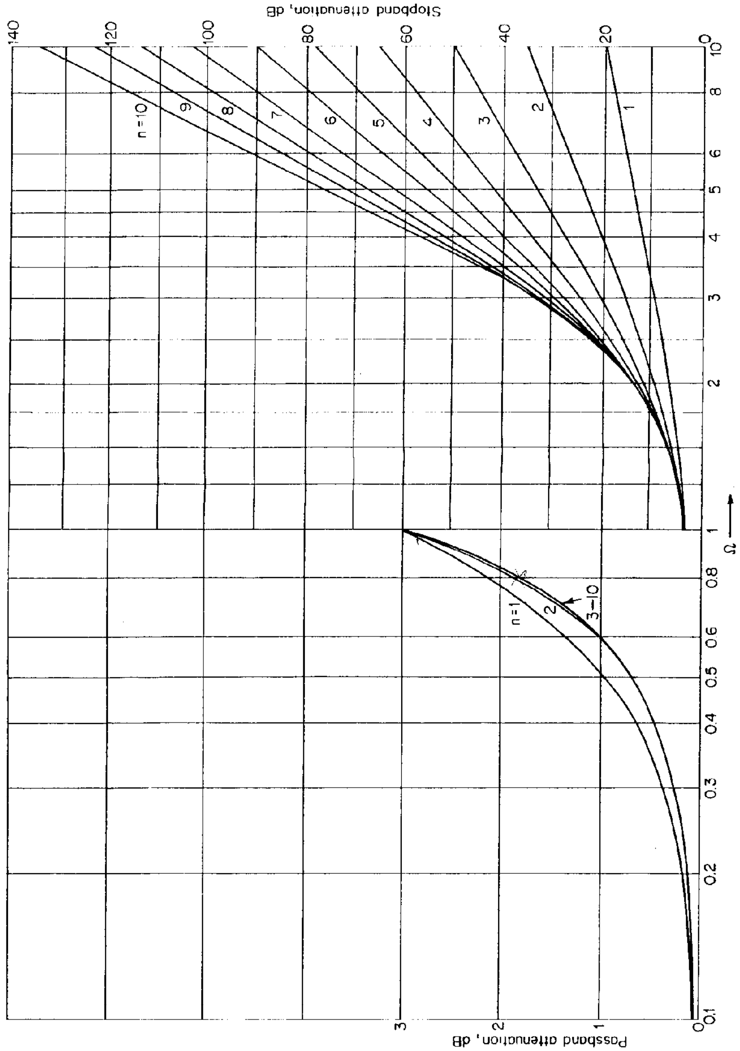


Fig. 2-56 Attenuation characteristics for maximally flat delay (Bessel) filters. *(From Anatol I. Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 1967. By permission of the publishers.)*

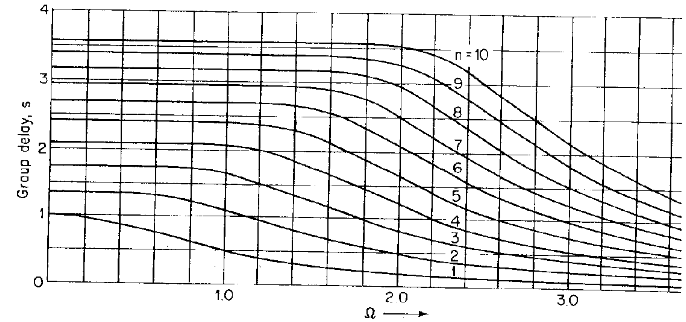


Fig. 2-57 Group-delay characteristics for maximally flat delay (Bessel) filters. *(Fr Anatol I. Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 19 By permission of the publishers.)*

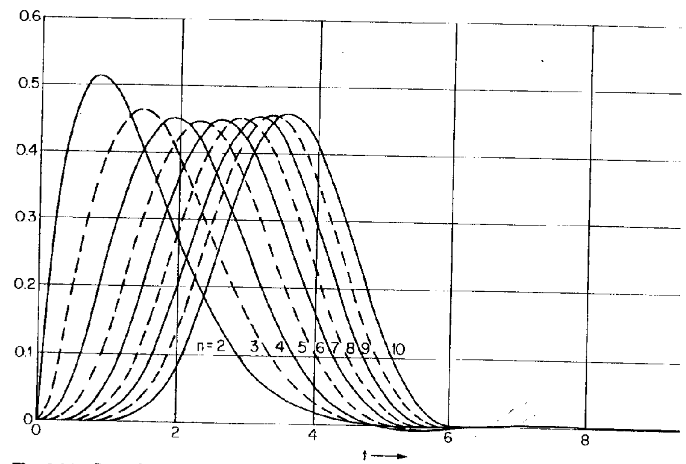


Fig. 2-58 Impulse response for maximally flat delay (Bessel) filters. *(From Anatol Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 1967. By permissio of the publishers.)*

A similar family of filters is the Gaussian type. However, the Gaussian ph response is not as linear as the Bessel for the same number of poles, and selectivity is not as sharp.

2.6 LINEAR PHASE WITH EQUIRIPPLE ERROR

The Chebyshev (equiripple amplitude) function is a better approximation an ideal amplitude curve than the Butterworth. Therefore, it stands to reas

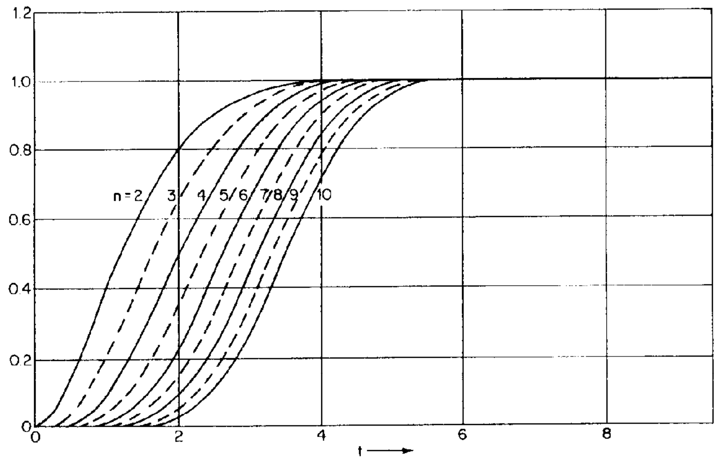


Fig. 2-59 Step response for maximally flat delay (Bessel) filters. *(From Anatol I. Zverev, Handbook of Filter Synthesis, John Wiley and Sons, Inc., New York, 1967. By permission of the publishers.)*

that an equiripple approximation of a linear phase will be more efficient than the Bessel family of filters.

Figure 2-60 illustrates how a linear phase can be approximated to within a given ripple of ϵ degrees. For the same n the equiripple-phase approximation results in a linear phase and consequently a constant delay over a larger interval than the Bessel approximation. Also the amplitude response is superior far from cutoff. In the transition region and below cutoff both approximations have nearly identical responses.

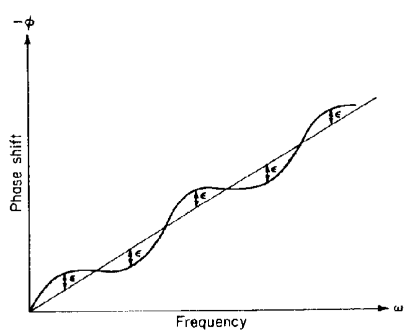


Fig. 2-60 Equiripple linear-phase approximation.

As the phase ripple ϵ is increased, the region of constant delay is exter farther into the stopband. However, the delay develops ripples. The step sponse has slightly more overshoot than Bessel filters.

A closed-form method for computation of the pole positions is not avail: The pole locations tabulated in chapter 12 were developed by iterative t niques. Values are provided for phase ripples of 0.05° and 0.5°, and the as: ated frequency and time-domain parameters are given in figures 2-61 thro 2-68.

2.7 TRANSITIONAL FILTERS

The Bessel filters discussed in section 2.5 have excellent transient prope but poor selectivity. Chebyshev filters, on the other hand, have steep rol characteristics but poor time-domain behavior. A transitional filter offers a c promise between a gaussian filter, which is similar to the Bessel family, Chebyshev filters.

Transitional filters have a near linear phase shift and smooth amplitude off in the passband. Outside the passband a sharp break in the amplitude cha teristics occurs. Beyond this breakpoint the attenuation increases quite abru in comparison with Bessel filters, especially for the higher n's.

In the tables in chapter 12 transitional filters are provided which have gaus characteristics to both 6 and 12 dB. The transient properties of the gaus to 6-dB filters are somewhat superior to those of the Butterworth family. Bey the 6-dB point, which occurs at approximately 1.5 rad/s, the attenuation cha teristics are nearly comparable with Butterworth filters. The gaussian 12-dB filters have time-domain parameters far superior to those of Butterw filters. However, the 12-dB breakpoint occurs at 2 rad/s, and the attenua characteristics beyond this point are inferior to those of Butterworth filters

The transitional filters tabulated in chapter 12 were generated by mathema techniques which involve interpolation of pole locations. Figures 2-69 thro 2-76 indicate the frequency and time-domain properties of both the gaus to 6-dB and gaussian to 12-dB transitional filters.

2.8 SYNCHRONOUSLY TUNED FILTERS

Synchronously tuned filters are the most basic filter type and are the eas to construct and align. They consist of identical multiple poles. A typical appl tion is in the case of a bandpass amplifier, where a number of stages are casca with each stage having the same center frequency and Q.

The attenuation of a synchronously tuned filter can be expressed as

$$A_{dB} = 10n \log[1+(2^{1/n}-1)\Omega^2] \qquad (2\text{-}$$

Equation (2-44) is normalized so that 3 dB of attenuation occurs at Ω = The individual section Q can be defined in terms of the composite circui requirement by the following relationship:

$$Q_{section} = Q_{overall}\sqrt{2^{1/n}-1} \qquad (2\text{-}$$

Alternately we can state that the 3-dB bandwidth of the individual secti is reduced by the shrinkage factor $(2^{1/n}-1)^{1/2}$. The individual section Q is l than the overall Q, whereas in the case of nonsynchronously tuned filters section Q's may be required to be much higher than the composite Q.