EELE445-14

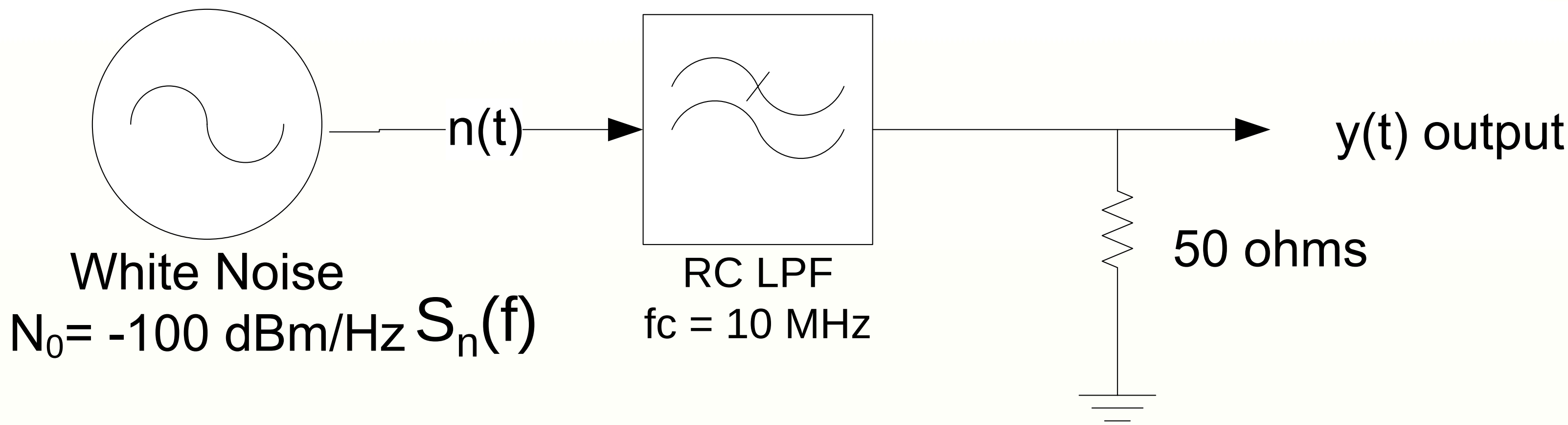
Lecture 11

Filter example,

Bandwidth definitions and BPSK example

MONTANA STATE UNIVERSITY

Example: White noise through filter



- Find $S_n(f)$ in Watts/Hz
- The equivalent noise bandwidth of the filter
- The output PSD, $S_y(f)$
- The total output noise power in dBm and in Watts
- The output rms noise voltage.
- The output Vp-p (assume 6σ noise)
- What would the filter BW have to be to reduce the noise power by 12 dB?

Bandlimited Signals

Bandlimited Waveforms

DEFINITION. A waveform $w(t)$ is said to be (absolutely) *bandlimited* to B hertz if

$$W(f) = \mathscr{F}[w(t)] = 0, \quad \text{for } |f| \geq B \tag{2–156}$$

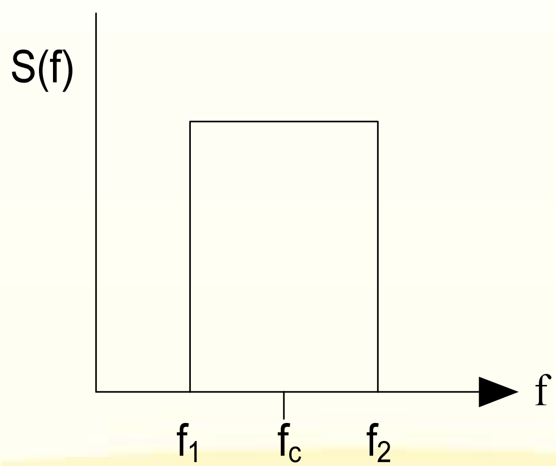
DEFINITION. A waveform $w(t)$ is (absolutely) *time limited* if

$$w(t) = 0, \quad \text{for } |t| > T \tag{2–157}$$

THEOREM. *An absolutely bandlimited waveform cannot be absolutely time limited, and vice versa.*

Bandwidth Definitions

1. ***<u>Absolute bandwidth</u>***: $B=f_2-f_1$, when the spectrum is zero outside the interval $f_1<f<f_2$ along the **<u>positive frequency axis.</u>** Example, white noise through an ideal bandpass filter
2. 3-dB bandwidth (or half-power bandwidth) is $B=f_2-f_1$, where for frequencies insde the band $f_1<f<f_2$, the power spectra, |S(f)|, fall now lower than ½ the maximum value of |S(f)|, and the maximum value occurs at a frequency inside the band.



3 - dB bandwidth (or half - power bandwidth) :

$$B_3 = f_2 - f_1$$

$$\frac{|S(f)|^2}{|S(f)|^2_{\max}} \geq \frac{1}{2} \; for \; \forall \; f, \; f_1 \leq f \leq f_2$$

Bandwidth Definitions

3. *Equivalent noise bandwidth* is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the power associated with the actual spectrum over positive frequencies. From Eq. (2–142), the PSD is proportional to the square of the magnitude of the spectrum. Let f_0 be the frequency at which the magnitude spectrum has a maximum; then the power in the equivalent rectangular band is proportional to

$$\text{equivalent power} = B_{eq}|H(f_0)|^2 \tag{2–190}$$

where B_{eq} is the equivalent bandwidth that is to be determined. The actual power for positive frequencies is proportional to

$$\text{actual power} = \int_0^\infty |H(f)|^2\, df \tag{2–191}$$

Setting Eq. (2–190) equal to Eq. (2–191), the formula for the *equivalent noise bandwidth* is

$$B_{eq} = \frac{1}{|H(f_0)|^2}\int_0^\infty |H(f)|^2\, df \tag{2–192}$$

Bandwidth Definitions

4. *Null-to-null bandwidth* (or zero-crossing bandwidth) is $f_2 - f_1$, where f_2 is the first null in the envelope of the magnitude spectrum above f_0 and, for bandpass systems, f_1 is the first null in the envelope below f_0, where f_0 is the frequency where the magnitude spectrum is a maximum.[†] For baseband systems, f_1 is usually zero.

5. *Bounded spectrum bandwidth* is $f_2 - f_1$ such that outside the band $f_1 < f < f_2$, the PSD, which is proportional to $|H(f)|^2$, must be down by at least a certain amount, say 50 dB, below the maximum value of the power spectral density.

6. *Power bandwidth* is $f_2 - f_1$, where $f_1 < f < f_2$ defines the frequency band in which 99% of the total power resides. This is similar to the FCC definition of *occupied bandwidth*, which states that the power above the upper band edge f_2 is $\frac{1}{2}$% and the power below the lower band edge is $\frac{1}{2}$%, leaving 99% of the total power within the occupied band (*FCC Rules and Regulations*, Sec. 2.202, Internet search: 47 CFR 2.202).

Bandwidth Definitions

7. *FCC bandwidth* is an authorized bandwidth parameter assigned by the FCC to specify the spectrum allowed in communication systems. When the FCC bandwidth parameter is substituted into the FCC formula, the minimum attenuation is given for the power level allowed in a 4-kHz band at the band edge with respect to the total average signal power. Sec. 21.106 (Internet search: 47 CFR 21.106) of the *FCC Rules and Regulations*: asserts, "For operating frequencies below 15 GHz, in any 4 kHz band, the center frequency of which is removed from the assigned frequency by more than 50 percent up to and including 250 percent of the authorized bandwidth, as specified by the following equation, but in no event less than 50 dB":

$$A = 35 + 0.8(P - 50) + 10 \log_{10}(B) \qquad (2\text{–}193)$$

(Attenuation greater than 80 dB is not required.) In this equation,

A = attenuation (in decibels) below the mean output power level,

P = percent removed from the carrier frequency,

and

B = authorized bandwidth in megahertz.

Bandwidth Definitions

The FCC definition (as well as many other legal rules and regulations) is somewhat obscure, but it will be interpreted in Example 2–18. It actually defines a spectral mask. That is, the spectrum of the signal must be less than or equal to the values given by this spectral mask at all frequencies. The FCC bandwidth parameter B is not consistent with the other definitions that are listed, in the sense that it is not proportional to $1/T$, the "signaling speed" of the corresponding signal [Amoroso, 1980]. Thus, the FCC bandwidth parameter B is a *legal* definition instead of an engineering definition. The *RMS bandwidth*, which is very useful in analytical problems, is defined in Chapter 6.

† In cases where there is no definite null in the magnitude spectrum, this definition is not applicable.

BPSK Signal

Example 2–22 Bandwidths for a BPSK Signal

A binary-phase-shift-keyed (BPSK) signal will be used to illustrate how the bandwidth is evaluated for the different definitions just given.

The BPSK signal is described by

$$s(t) = m(t)\cos\omega_c t \tag{2–194}$$

where $\omega_c = 2\pi f_c$, f_c being the carrier frequency (hertz), and $m(t)$ is a serial binary (± 1 values) modulating waveform originating from a digital information source such as a digital computer, as illustrated in Fig. 2–23a. Let us evaluate the spectrum of $s(t)$ for the worst case (the widest bandwidth).

The worst-case (widest-bandwidth) spectrum occurs when the digital modulating waveform has transitions that occur most often. In this case $m(t)$ would be a square wave, as shown in Fig. 2–23a. Here a binary 1 is represented by $+1$ V and a binary 0 by -1 V, and the signaling rate is $R = 1/T_b$ bits/s. The power spectrum of the square-wave modulation can be evaluated by using Fourier series analysis. Equations (2–126) and (2–120) give

$$\mathcal{P}_m(f) = \sum_{n=-\infty}^{\infty} |c_n|^2\,\delta(f - nf_0) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{\sin(n\pi/2)}{n\pi/2}\right]^2 \delta\left(f - n\frac{R}{2}\right) \tag{2–195}$$

BPSK Signal

Example 2–22 BANDWIDTHS FOR A BPSK SIGNAL

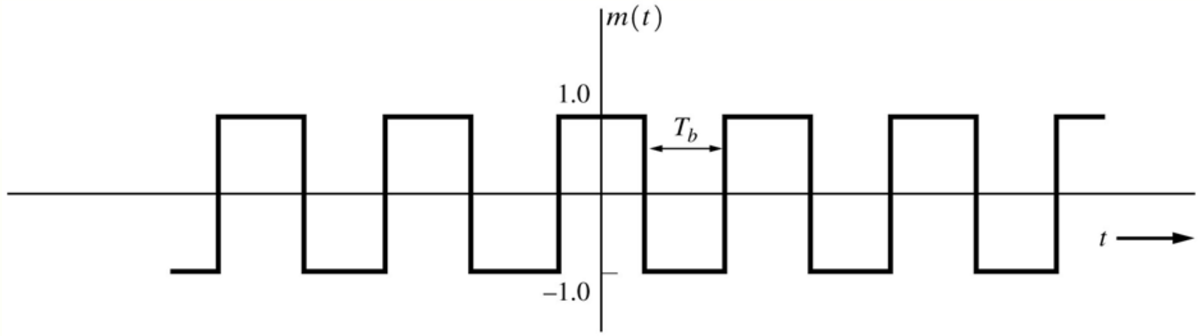
where $f_0 = 1/(2T_b) = R/2$. The PSD of $s(t)$ can be expressed in terms of the PSD of $m(t)$ by evaluating the autocorrelation of $s(t)$—that is,

$$\begin{aligned} R_s(\tau) &= \langle s(t)s(t+\tau)\rangle \\ &= \langle m(t)m(t+\tau)\cos\omega_c t\cos\omega_c(t+\tau)\rangle \\ &= \tfrac{1}{2}\langle m(t)m(t+\tau)\rangle\cos\omega_c\tau + \tfrac{1}{2}\langle m(t)m(t+\tau)\cos(2\omega_c t+\omega_c\tau)\rangle \end{aligned}$$

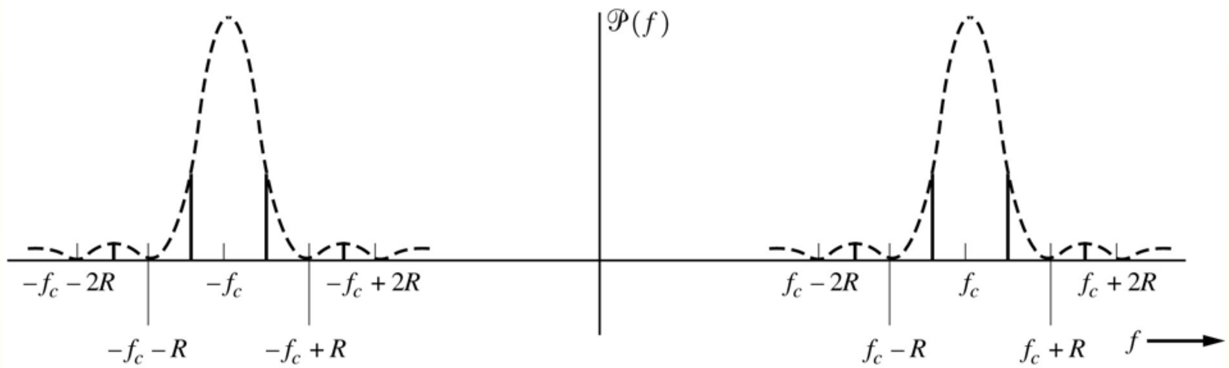
or

$$R_s(\tau) = \tfrac{1}{2}R_m(\tau)\cos\omega_c\tau + \tfrac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} m(t)m(t+\tau)\cos(2\omega_c t+\omega_c\tau)\,dt \qquad (2\text{–}196)$$

Figure 2–23 Spectrum of a BPSK signal.



(a) Digital Modulating Waveform



(b) Resulting BPSK Spectrum

BPSK Signal

Example 2–22 BANDWIDTHS FOR A BPSK SIGNAL

The integral is negligible because $m(t)m(t+\tau)$ is constant over small time intervals, but cos $(2\omega_c t + \omega_c\tau)$ has many cycles of oscillation, since $f_c \gg R$.† Any small area that is accumulated by the integral becomes negligible when divided by T, $T \to \infty$. Thus,

$$R_s(\tau) = \tfrac{1}{2} R_m(\tau) \cos \omega_c\tau \tag{2–197}$$

The PSD is obtained by taking the Fourier transform of both sides of Eq. (2–197). Using the real-signal frequency transform theorem (Table 2–1), we get

$$\mathcal{P}_s(f) = \tfrac{1}{4}\left[\mathcal{P}_m(f - f_c) + \mathcal{P}_m(f + f_c)\right] \tag{2–198}$$

Substituting Eq. (2–195) into Eq. (2–198), we obtain the PSD for the BPSK signal:

$$\mathcal{P}_s(f) = \tfrac{1}{4} \sum_{\substack{n=-\infty \\ n\neq\infty}}^{\infty} \left[\frac{\sin(n\pi/2)}{n\pi/2}\right]^2 \times \{\delta[f - f_c - n(R/2)] + \delta[f + f_c - n(R/2)]\} \tag{2–199}$$

This spectrum is also shown in Fig. 2–23b. See Example 2_22.m for a plot of Eq. (2–199).

The spectral shape that results from utilizing this worst-case deterministic modulation is essentially the same as that obtained when random data are used; however, for the random case,

† This is a consequence of the Riemann-Lebesque lemma from integral calculus [Olmsted, 1961].

BPSK Signal

Example 2–22 BANDWIDTHS FOR A BPSK SIGNAL

the spectrum is continuous. The result for the random data case, as worked out in Chapter 3 where $\mathcal{P}_m(f)$ is given by Eq. (3–41), is

$$\mathcal{P}(f) = \frac{1}{4} T_b \left[\frac{\sin \pi T_b(f - f_c)}{\pi T_b(f - f_c)} \right]^2 + \frac{1}{4} T_b \left[\frac{\sin \pi T_b(f + f_c)}{\pi T_b(f + f_c)} \right]^2 \qquad (2\text{–}200)$$

when the data rate is $R = 1/T_b$ bits/sec. This is shown by the dashed curve in Fig. 2–23b.

The preceding derivation demonstrates that we can often use (deterministic) square-wave test signals to help us analyze a digital communication system, instead of using a more complicated random-data model.

The bandwidth for the BPSK signal will now be evaluated for each of the bandwidth definitions given previously. To accomplish this, the shape of the PSD for the positive frequencies is needed. From Eq. (2–200), it is

$$\mathcal{P}(f) = \left[\frac{\sin \pi T_b(f - f_c)}{\pi T_b(f - f_c)} \right]^2 \qquad (2\text{–}201)$$

BPSK Signal

Example 2–22 BANDWIDTHS FOR A BPSK SIGNAL

Substituting Eq. (2–201) into the definitions, we obtain the resulting BPSK bandwidths as shown in Table 2–4, except for the FCC bandwidth.

The relationship between the spectrum and the FCC bandwidth parameter is a little more tricky to evaluate. To do that, we need to evaluate the decibel attenuation

$$A(f) = -10 \log_{10}\left[\frac{P_{4\text{kHz}}(f)}{P_{\text{total}}}\right] \qquad (2\text{–}202)$$

where $P_{4\text{kHz}}(f)$ is the power in a 4-kHz band centered at frequency f and P_{total} is the total signal power. The power in a 4-kHz band (assuming that the PSD is approximately constant across the 4-kHz bandwidth) is

$$P_{4\text{kHz}}(f) = 4000\,\mathcal{P}(f) \qquad (2\text{–}203)$$

and, using the definition of equivalent bandwidth, we find that the total power is

$$P_{\text{total}} = B_{\text{eq}}P(f_c) \qquad (2\text{–}204)$$

BPSK Signal

TABLE 2–4 BANDWIDTHS FOR BPSK SIGNALING WHERE THE BIT RATE IS $R = 1/T_b$ BITS/S.

Definition Used	Bandwidth	Bandwidths (kHz) for $R = 9{,}600$ bits/s
1. Absolute bandwidth	∞	∞
2. 3-dB bandwidth	$0.88R$	8.45
3. Equivalent noise bandwidth	$1.00R$	9.60
4. Null-to-null bandwidth	$2.00R$	19.20
5. Bounded spectrum bandwidth (50 dB)	$201.04R$	1,930.0
6. Power bandwidth	$20.56R$	197.4

Example 2–22 BANDWIDTHS FOR A BPSK SIGNAL

where the spectrum has a maximum value at $f = f_c$. With the use of these two equations, Eq. (2–202) becomes

$$A(f) = -10 \log_{10}\left[\frac{4000\mathcal{P}(f)}{B_{\text{eq}}\mathcal{P}(f_c)}\right] \tag{2–205}$$

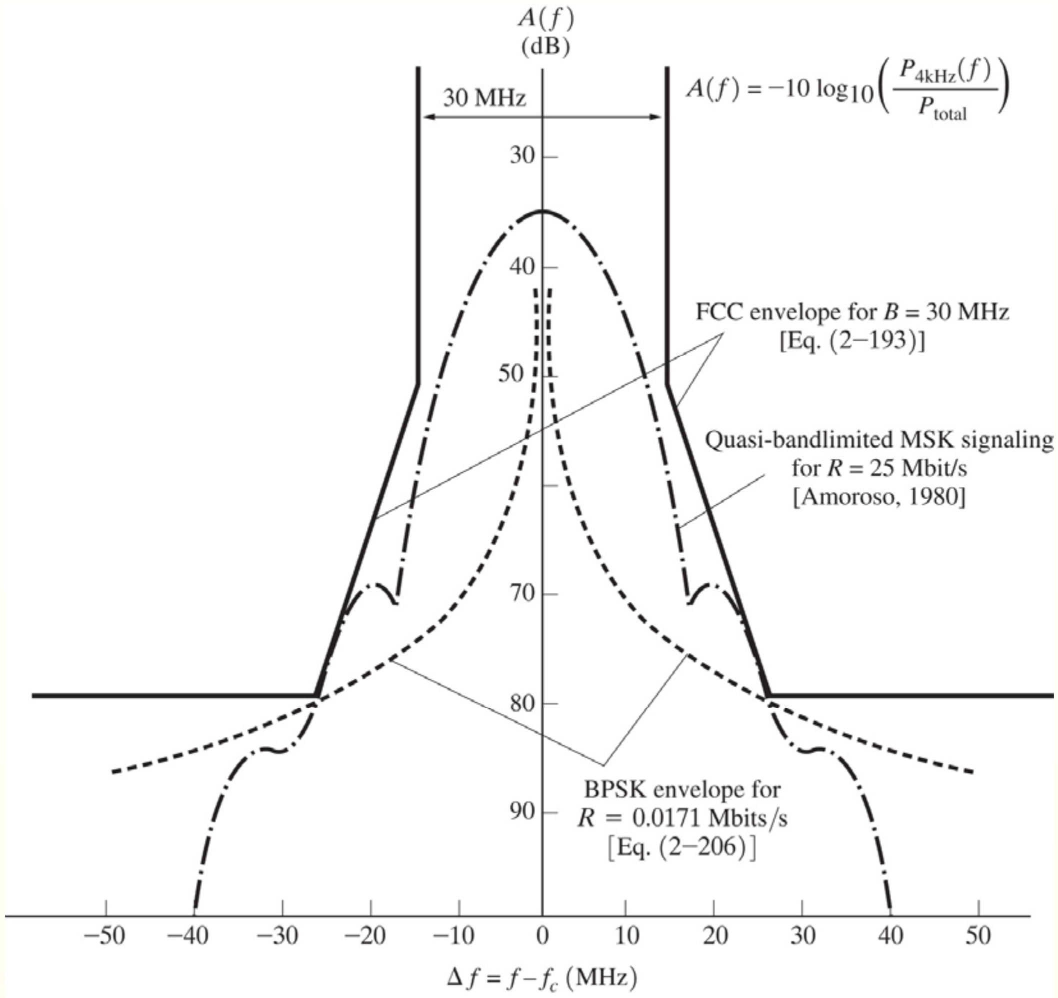
where $A(f)$ is the decibel attenuation of power measured in a 4-kHz band at frequency f compared with the total average power level of the signal. For the case of BPSK signaling, using Eq. (2–201), we find that the decibel attenuation is

$$A(f) = -10 \log_{10}\left\{\frac{4000}{R}\left[\frac{\sin \pi T_b(f - f_c)}{\pi T_b(f - f_c)}\right]^2\right\} \tag{2–206}$$

where $R = 1/T_b$ is the data rate. If we attempt to find the value of R such that $A(f)$ will fall below the specified FCC spectral envelope shown in Fig. 2–24 ($B = 30\,\text{MHz}$), we will find that R is so small that there will be numerous zeros in the $(\sin x)/x$ function of Eq. (2–206) within the desired frequency range, $-50\,\text{MHz} < (f - f_c) < 50\,\text{MHz}$. This is difficult to plot, so by

BPSK Signal

Figure 2–24 FCC-allowed envelope for B = 30 MHz.



Example 2–22 BANDWIDTHS FOR A BPSK SIGNAL

replacing $\sin \pi T(f - f_c)$ with its maximum value (unity) we plot the envelope of $A(f)$ instead. The resulting BPSK envelope curve for the decibel attenuation is shown in Fig. 2–24, where $R = 0.0171$ Mbits/s.

It is obvious that the data rate allowed for a BPSK signal to meet the FCC $B = 30$-MHz specification is ridiculously low, because the FCC bandwidth parameter specifies an almost absolutely bandlimited spectrum. To signal with a reasonable data rate, the pulse shape used in the transmitted signal must be modified from a rectangular shape (that gives the BPSK signal) to a rounded pulse such that the bandwidth of the transmitted signal will be almost absolutely bandlimited. Recalling our study of the sampling theorem, we realize that $(\sin x)/x$ pulse shapes are prime candidates, since they have an absolutely bandlimited spectrum. However, the $(\sin x)/x$ pulses are not absolutely timelimited, so that this exact pulse shape cannot be used. Frank Amoroso and others have studied the problem, and a quasi-bandlimited version of the $(\sin x)/x$ pulse shape has been proposed [Amoroso, 1980]. The decibel attenuation curve for this type of signaling, shown in Fig. 2–24, is seen to fit very nicely under the allowed FCC spectral envelope curve for the case of $R = 25$ Mbits/s. The allowable data rate of 25 Mbits/s is a fantastic improvement over the $R = 0.0171$ Mbits/s that was allowed for BPSK. It is also interesting to note that analog pulse shapes [$(\sin x)/x$ type] are required instead of a digital (rectangular) pulse shape, which is another way of saying that *it is vital for digital communication engineers to be able to analyze and design analog systems as well as digital systems.*

Sampling
PAM- Pulse Amplitude Modulation
PCM- Pulse Code Modulation

EELE445-14

Lecture 12

EELE445-14

Lecture 12

Sampling

MONTANA STATE UNIVERSITY

Mountains & Minds

Sampling – The Cardinal Series

Sampling Theorem: ***Any physical waveform may be represented over the interval*** $-\infty \le t \le \infty$ *by*

$$w(t) = \sum_{n=-\infty}^{n=\infty} a_n \frac{\sin\{\pi f_s[t - (n/f_s)]\}}{\pi f_s[t - (n/f_s)]}$$

$$a_n = f_s \int_{-\infty}^{\infty} w(t) \frac{\sin\{\pi f_s[t - (n/f_s)]\}}{\pi f_s[t - (n/f_s)]} dt$$

where f_s is a parameter assigned some convenient value greater than zero

MONTANA STATE UNIVERSITY

Mountains & Minds

Sampling – The Cardinal Series

if $w(t)$ is bandlimited to B hertz and $f_s \geq 2B$, then Eq. (2–158) becomes the sampling function representation, where

$$a_n = w(n/f_s) \tag{2–160}$$

That is, for $f_s \geq 2B$, the orthogonal series coefficients are simply the values of the waveform that are obtained when the waveform is sampled every $1/f_s$ seconds.

$$\left(f_s\right)_{\min} = 2B$$ **Nyquist Frequency**

Sampling – The Cardinal Series

$$\varphi_n(t) = \frac{\sin\{\pi f_s[t - (n/f_s)]\}}{\pi f_s[t-(n/f_s)]} \tag{2–161}$$

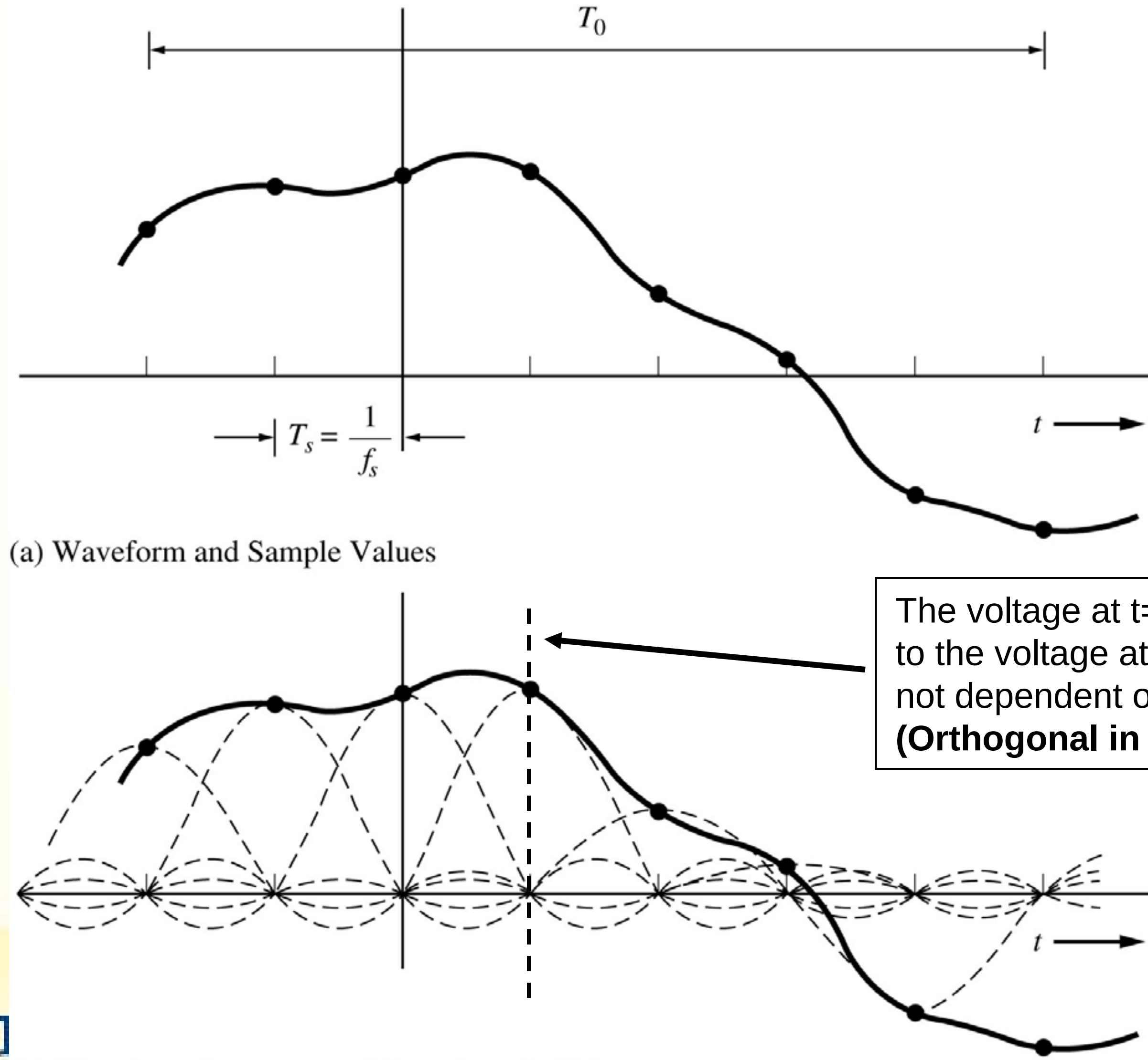
$$w(t) \approx \sum_{n=n_1}^{n=n_1+N} a_n \varphi_n(t) \tag{2–169}$$

$$N = \frac{T_0}{1/f_s} = f_s T_0 \geq 2BT_0 \tag{2–170}$$

and there are N orthogonal functions in the reconstruction algorithm. We can say that N is the number of dimensions needed to reconstruct the T_0-second approximation of the waveform.

Figure 2–17 Sampling theorem.

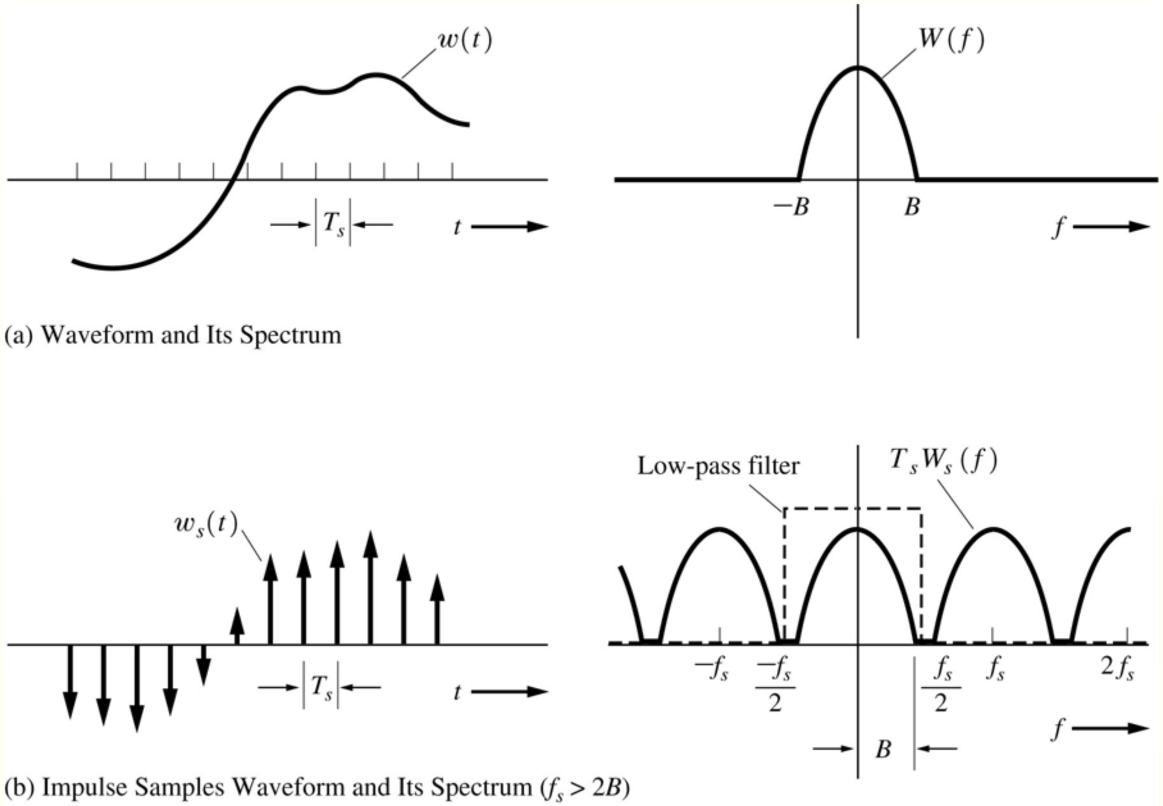
Sampling – The Cardinal Series



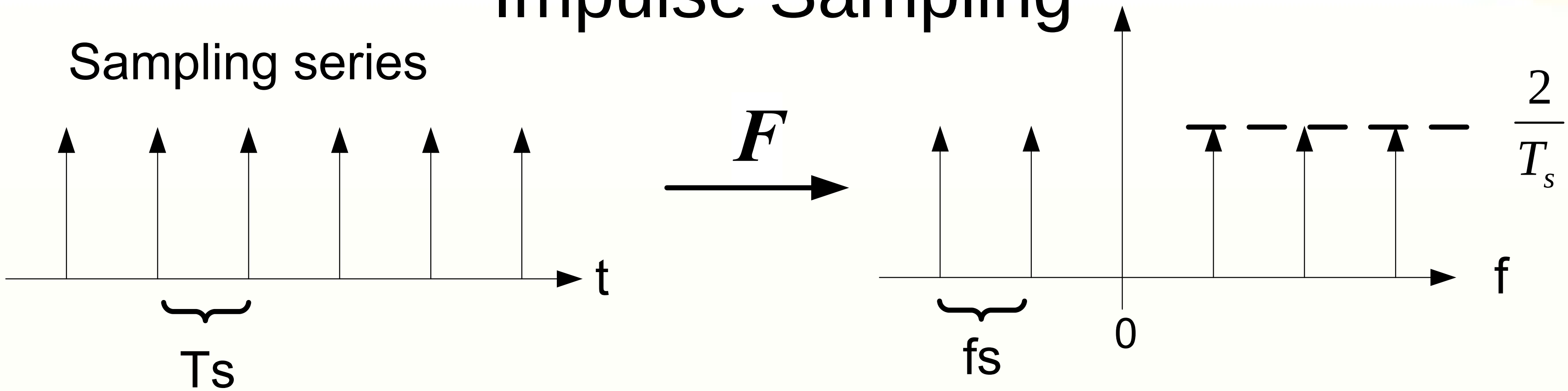
(b) Waveform Reconstructed from Sample Values

The voltage at $t=nT_s$ is only due to the voltage at sample nT_s and is not dependent on any other sample. **(Orthogonal in time!)**

Figure 2–18 Impulse sampling.



Impulse Sampling

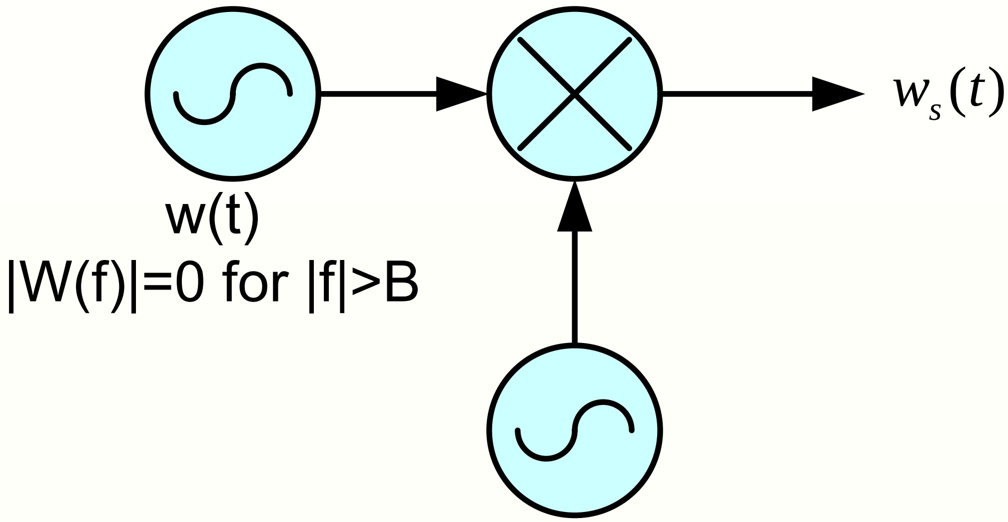


$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \Rightarrow D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_s t + \varphi_n)$$

$$\varphi_n = 0 \qquad \omega_s = \frac{2\pi}{T_s} \qquad D_0 = \frac{1}{T_s} \qquad D_n = \frac{2}{T_s}$$

$$\delta_{T_s}(t) = \frac{1}{T_s}\left[1 + 2\cos(\omega_s t) + 2\cos(2\omega_s t) + 2\cos(3\omega_s t) + \cdots\right]$$

Impulse Sampling



Impulse sampling series $\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

$$w_s(t) = w(t)\delta_{T_s}(t) = \frac{w(t)}{T_s}\left[1 + 2\cos(\omega_s t) + 2\cos(2\omega_s t) + 2\cos(3\omega_s t) + \cdots\right]$$

$$W_s(f) = F\left[w_s(t)\right] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} W(f - nf_s)$$

Impulse Sampling- text

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} w(nT_s)\delta(t - nT_s) \qquad eq2-171$$

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

$$W_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} W(f - nf_s)$$

Impulse Sampling

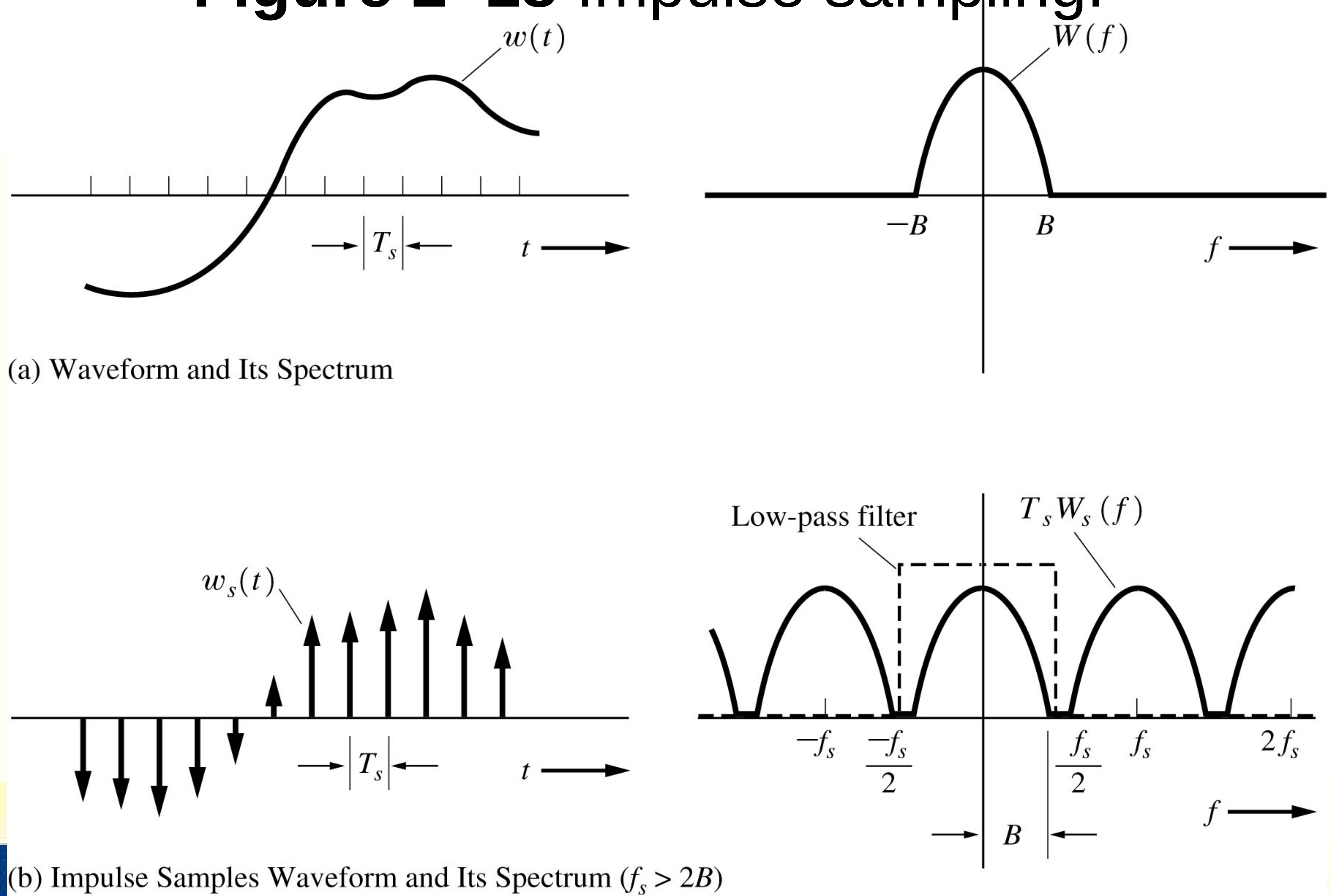
The spectrum of the impuse sampled signal is the spectrum of the unsampled signal that is repeated every f_s Hz, where f_s is the sampling frequency or rate (samples/sec). This is one of the basic principles of digital signal processing, DSP.

Note:

This technique of impulse sampling is often used to translate the spectrum of a signal to another frequency band that is centered on a harmonic of the sampling frequency, f_s.

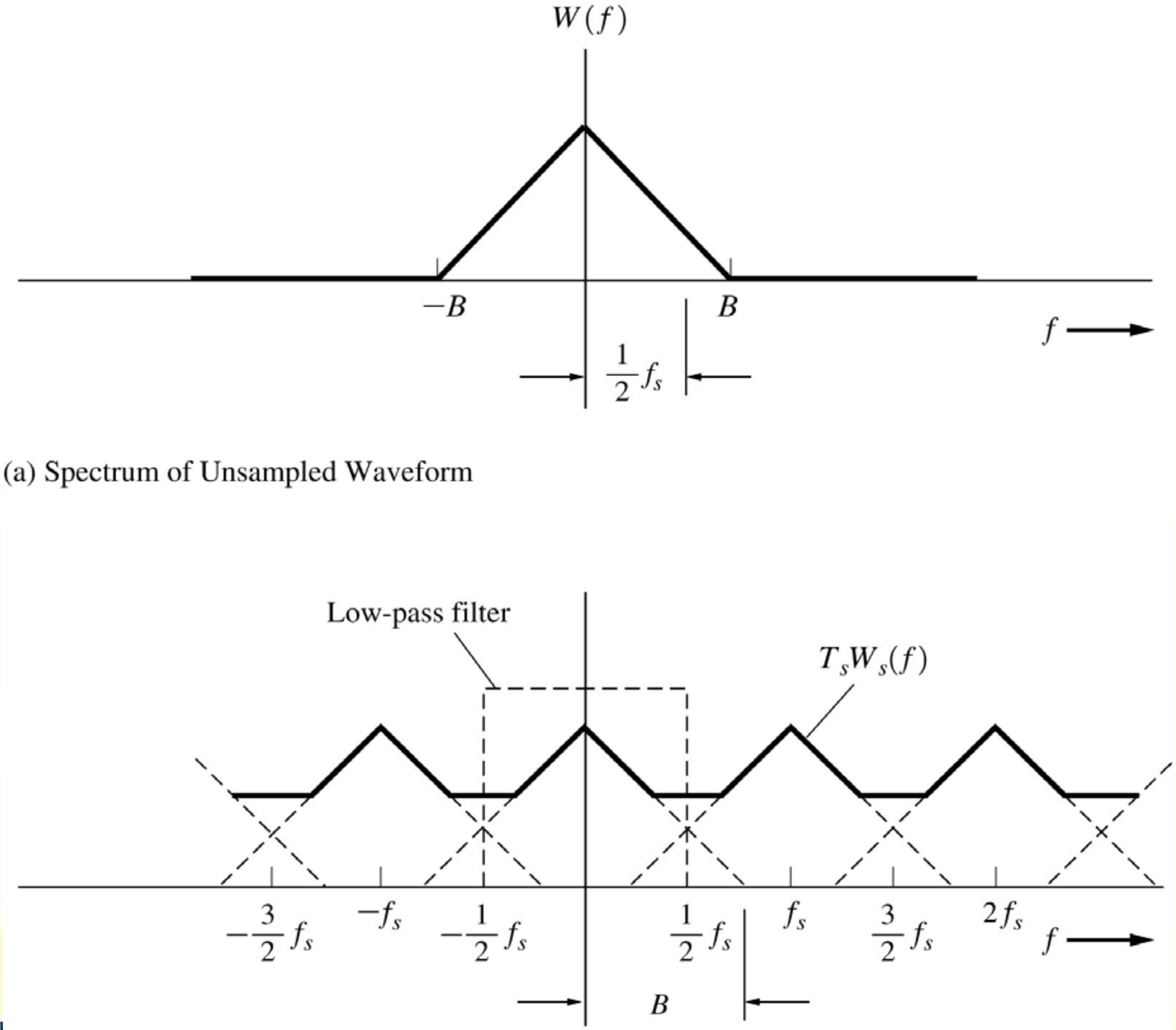
If f_s>=2B, (see fig 2-18), the replicated spectra around each harmonic of f_s do not overlap, and the original spectrum can be regenerated with an ideal LPF with a cutoff of $f_s/2$.

Figure 2–18 Impulse sampling.



Impulse Sampling

Figure 2–19 Undersampling and aliasing.

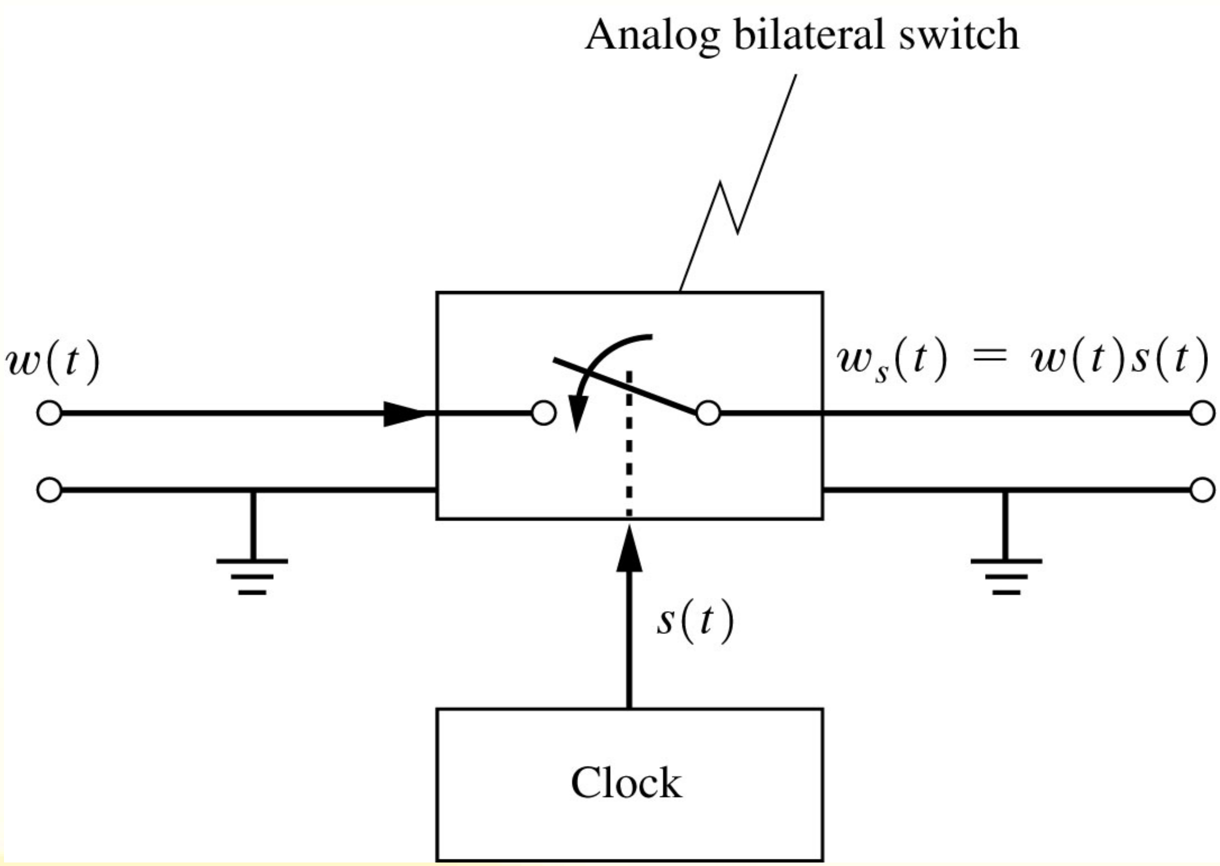


(a) Spectrum of Unsampled Waveform

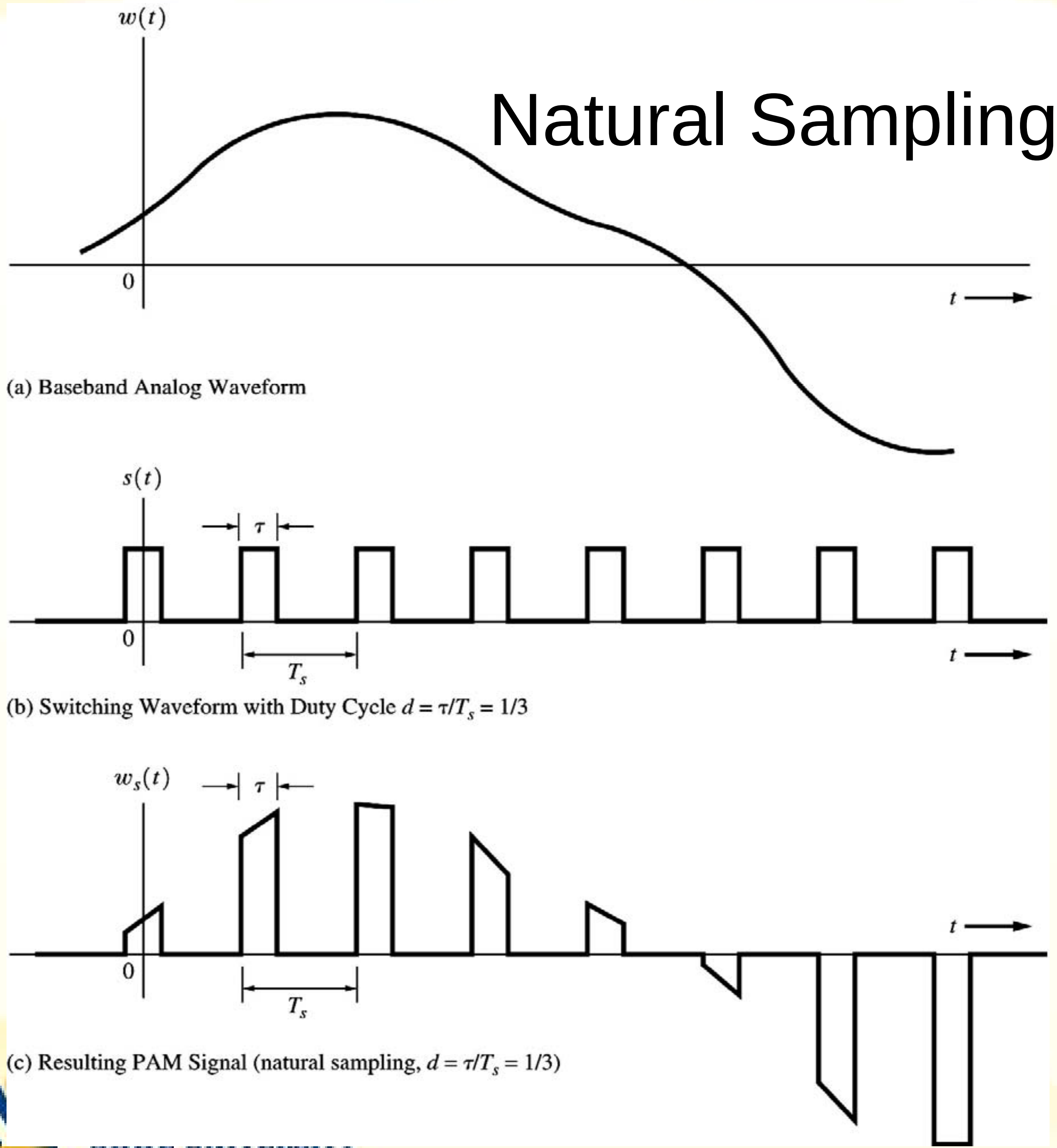
(b) Spectrum of Impulse Sampled Waveform ($f_s < 2B$)

Natural Sampling

Generation of PAM with natural sampling (gating).

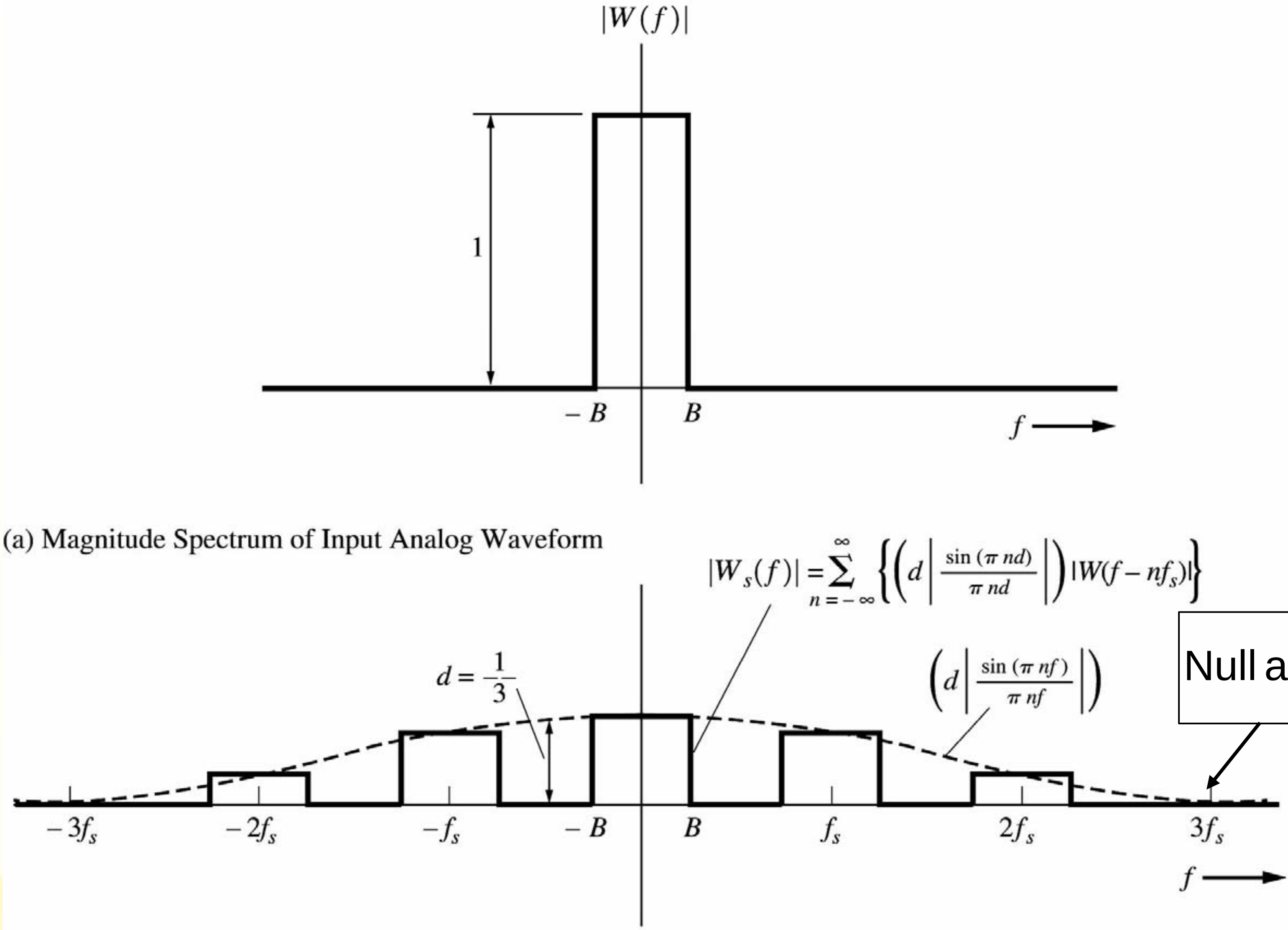


Natural Sampling



Duty cycle =1/3

Natural Sampling



(b) Magnitude Spectrum of PAM (natural sampling) with $d = 1/3$ and $f_s = 4$ B

Figure 3–4 Demodulation of a PAM signal (naturally sampled).

n^{th} Nyquist region recovery

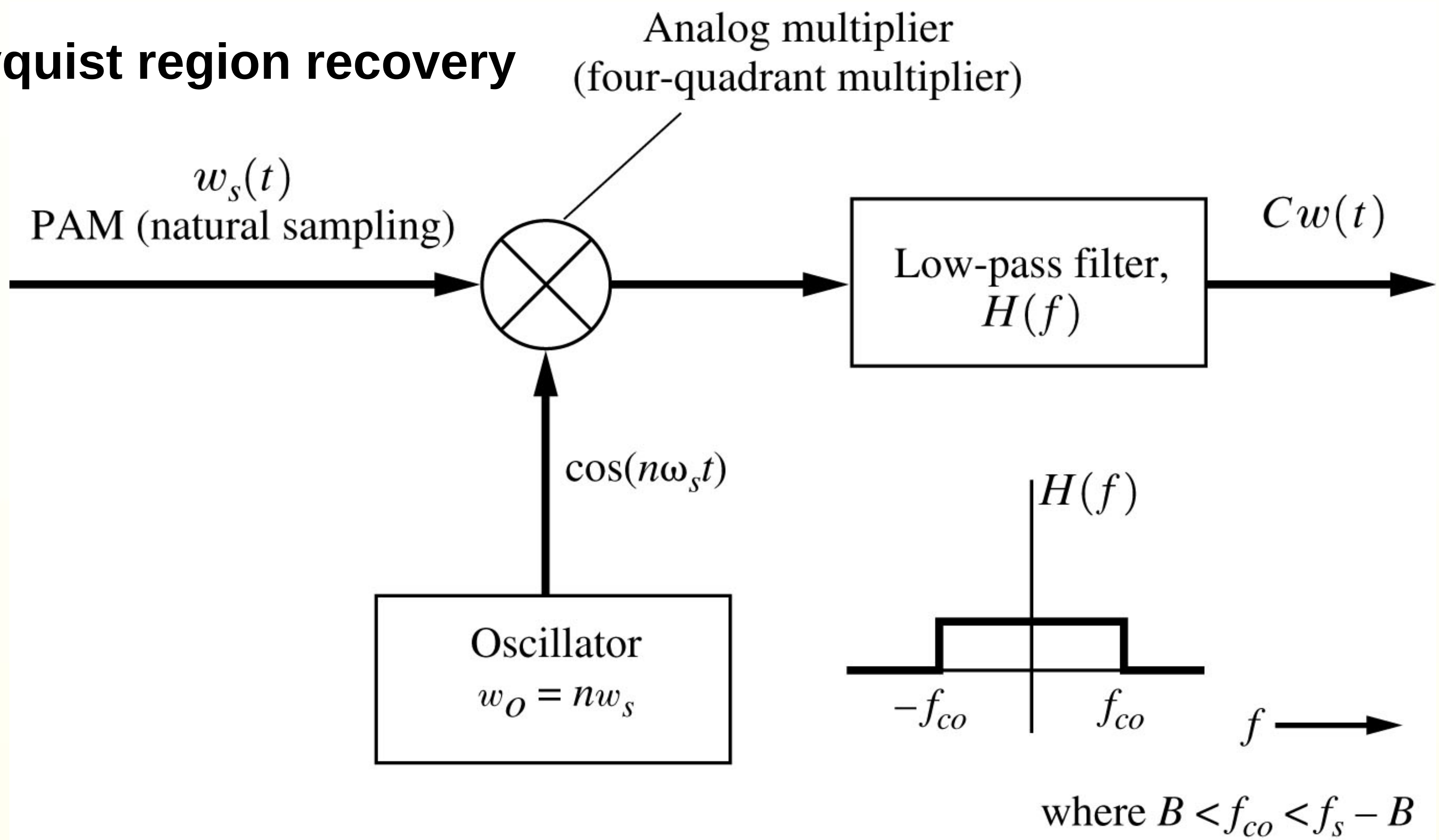


Figure 3–5 PAM signal with flat-top sampling.

Impulse sample and hold

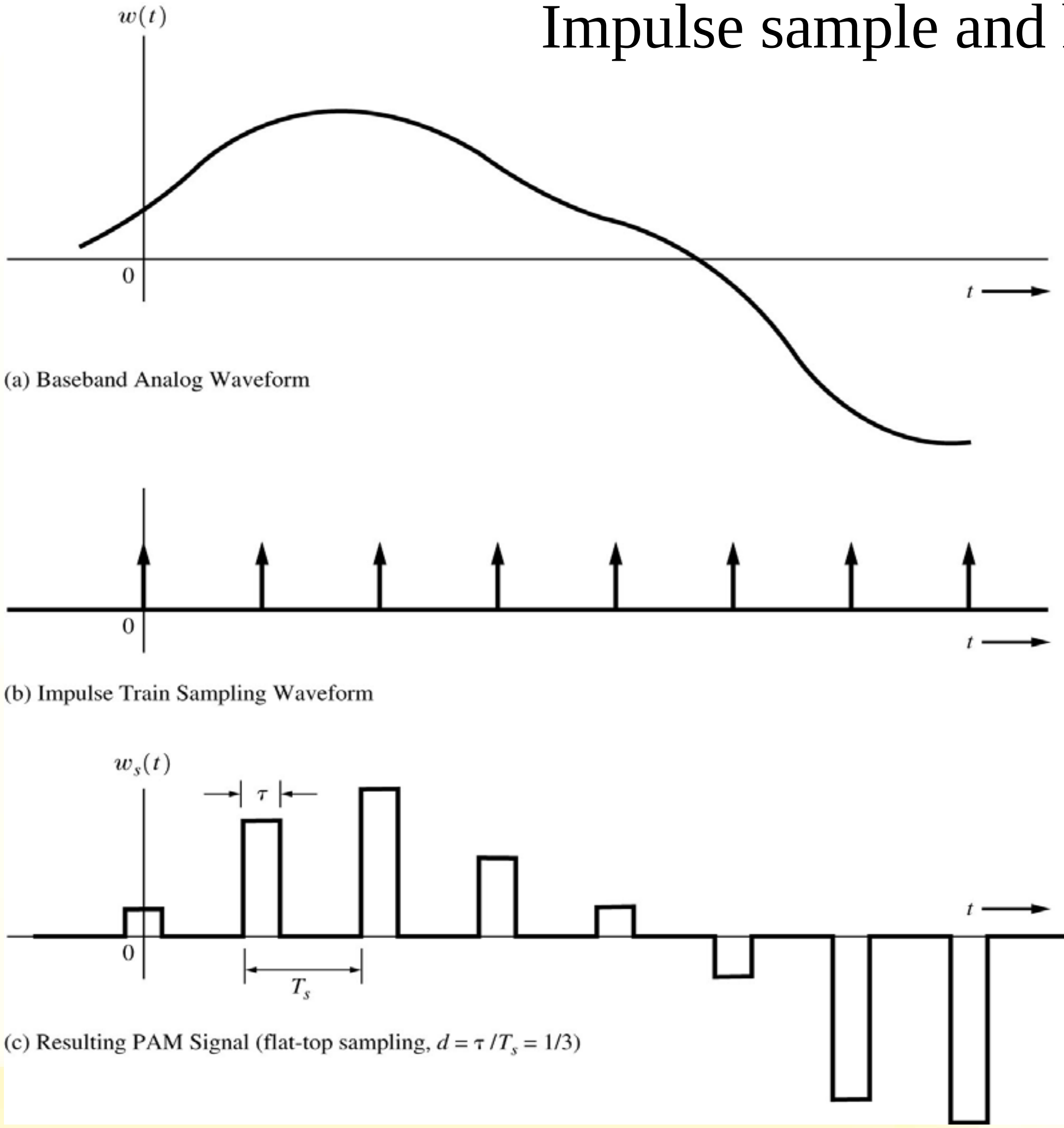
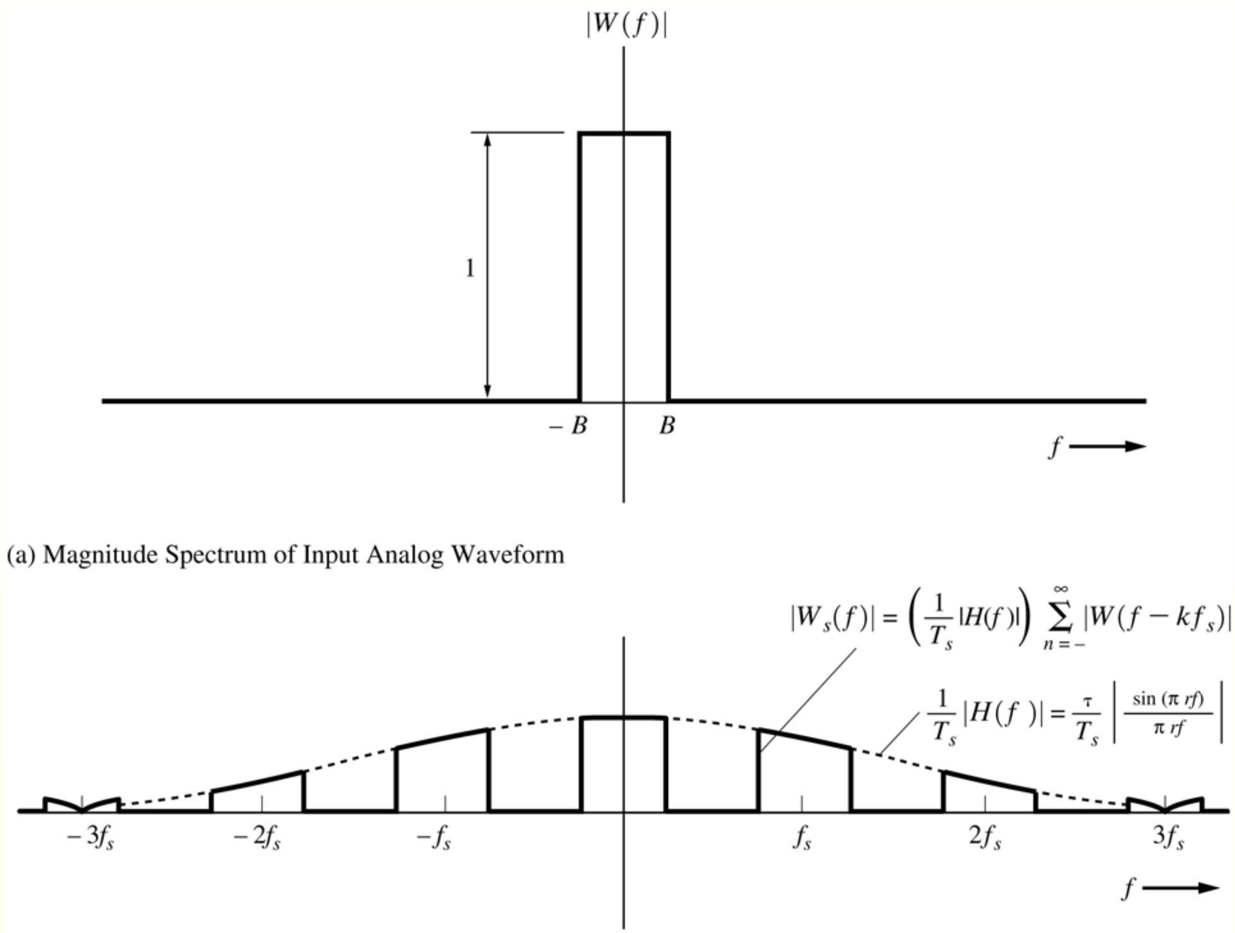


Figure 3–6 Spectrum of a PAM waveform with flat-top sampling.



(a) Magnitude Spectrum of Input Analog Waveform

(b) Magnitude Spectrum of PAM (flat-top sampling), $\tau/T_s = 1/3$ and $f_s = 4B$

Figure 3–7 PCM trasmission system.

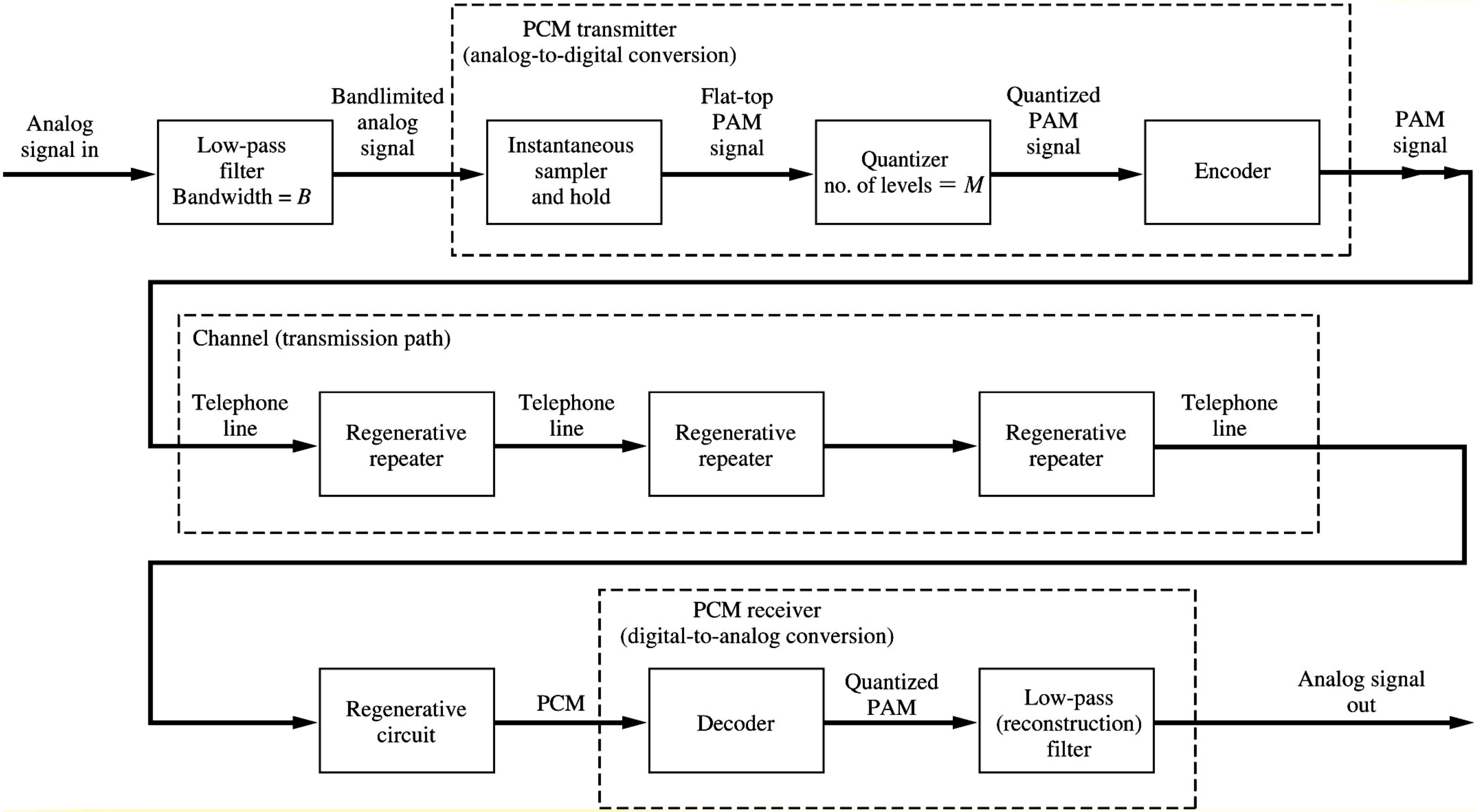
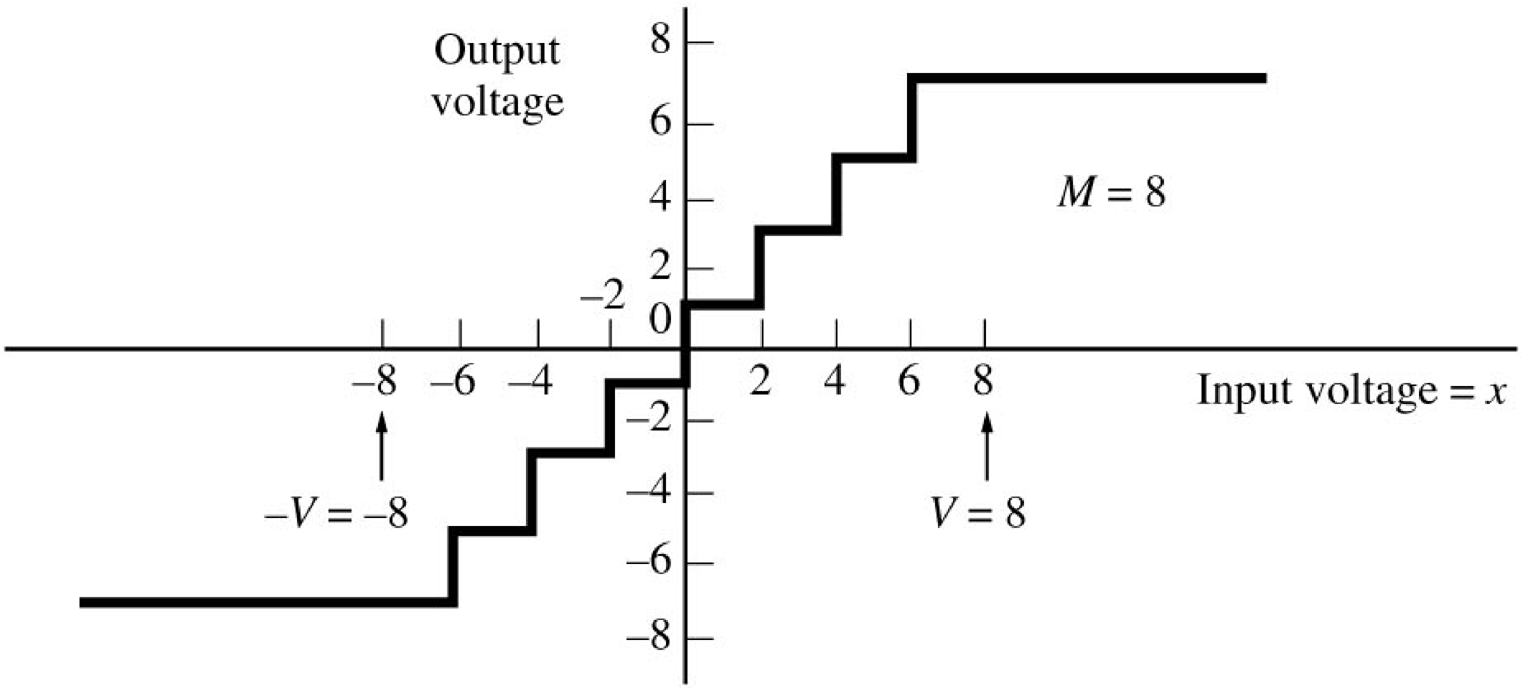
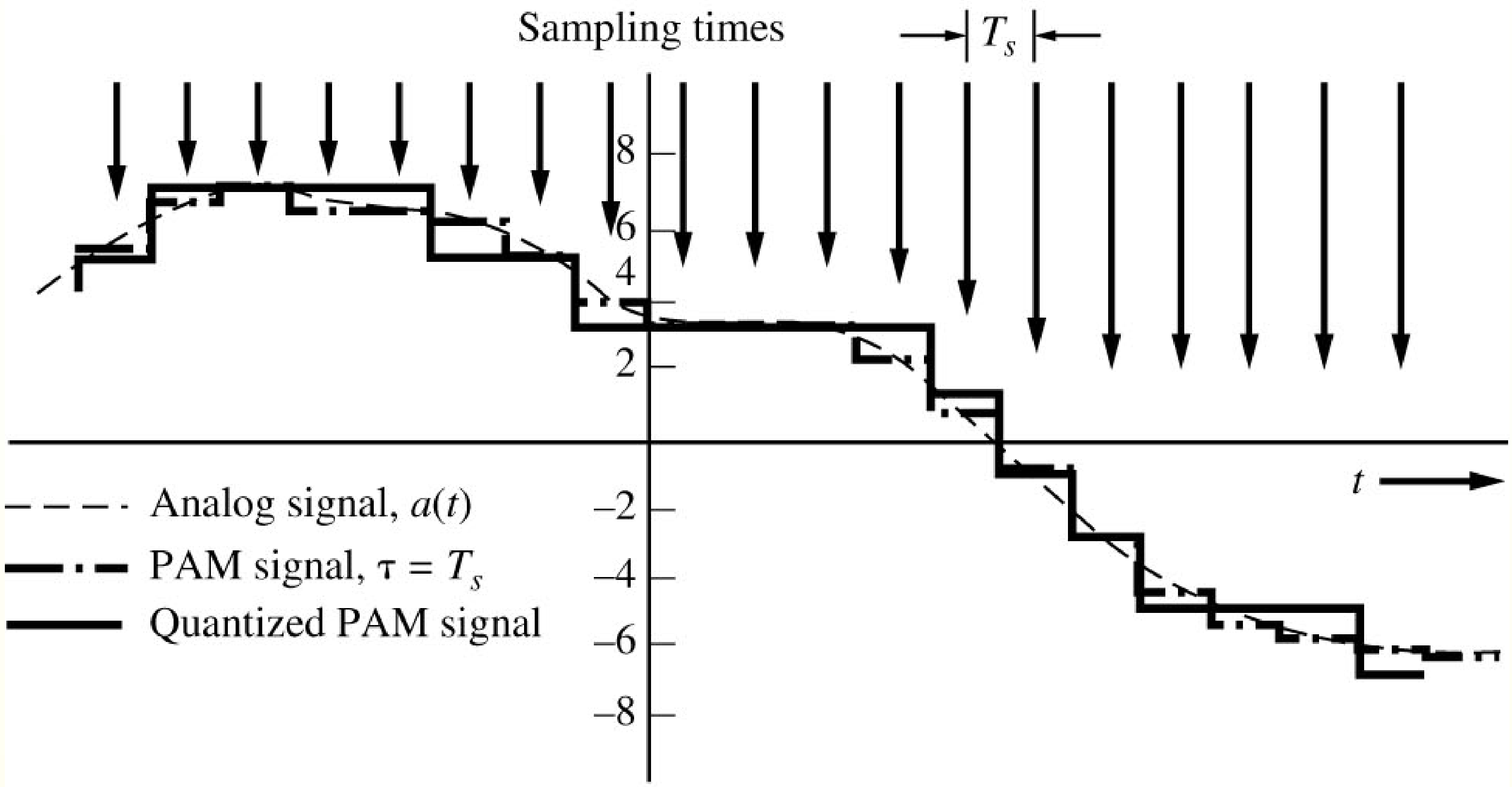


Figure 3–8 Illustration of waveforms in a PCM system.



(a) Quantizer Output-Input Characteristics

Figure 3–8 Illustration of waveforms in a PCM system.



(b) Analog Signal, Flat-top PAM Signal, and Quantized PAM Signal

Figure 3–8 Illustration of waveforms in a PCM system.

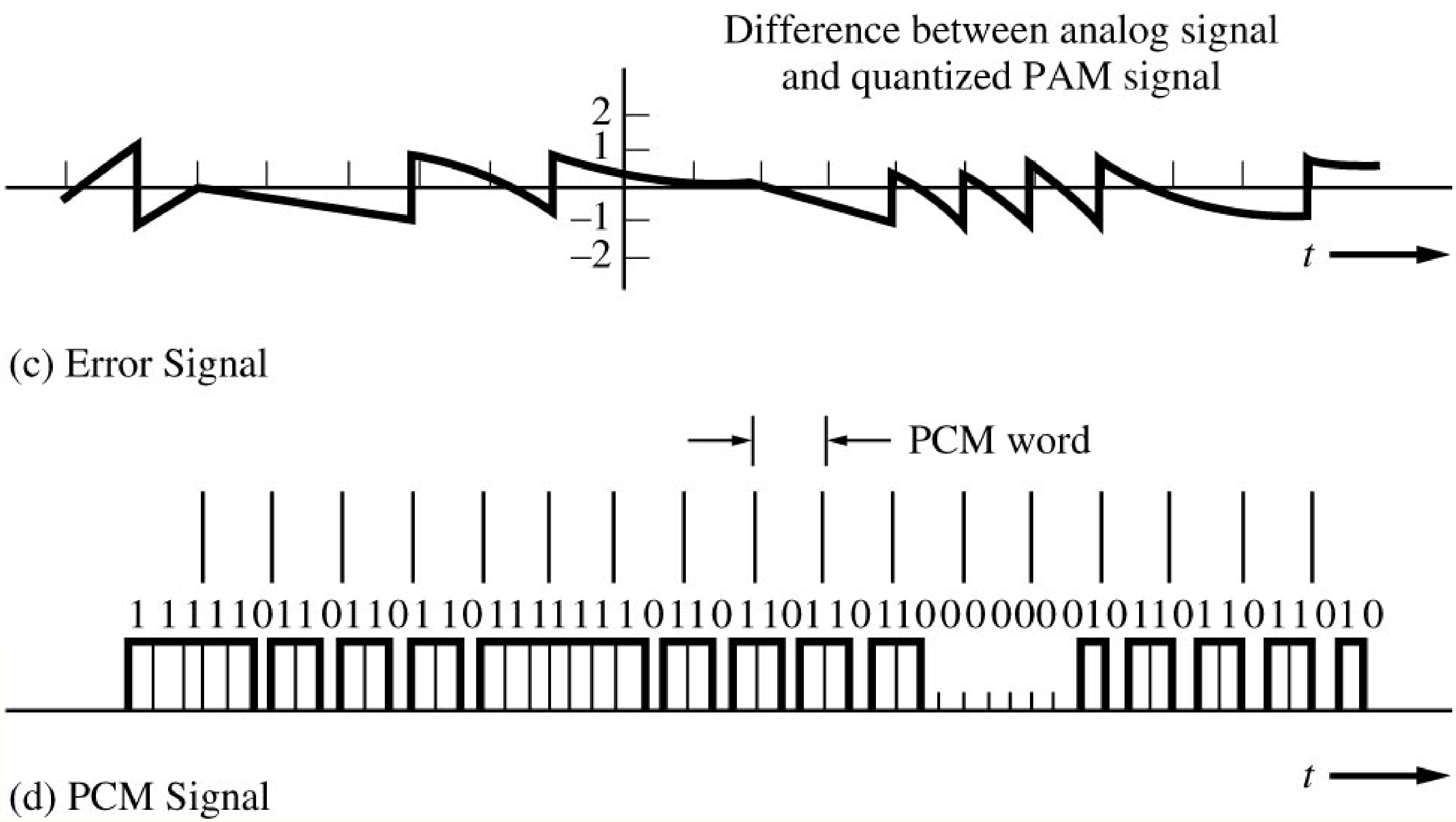
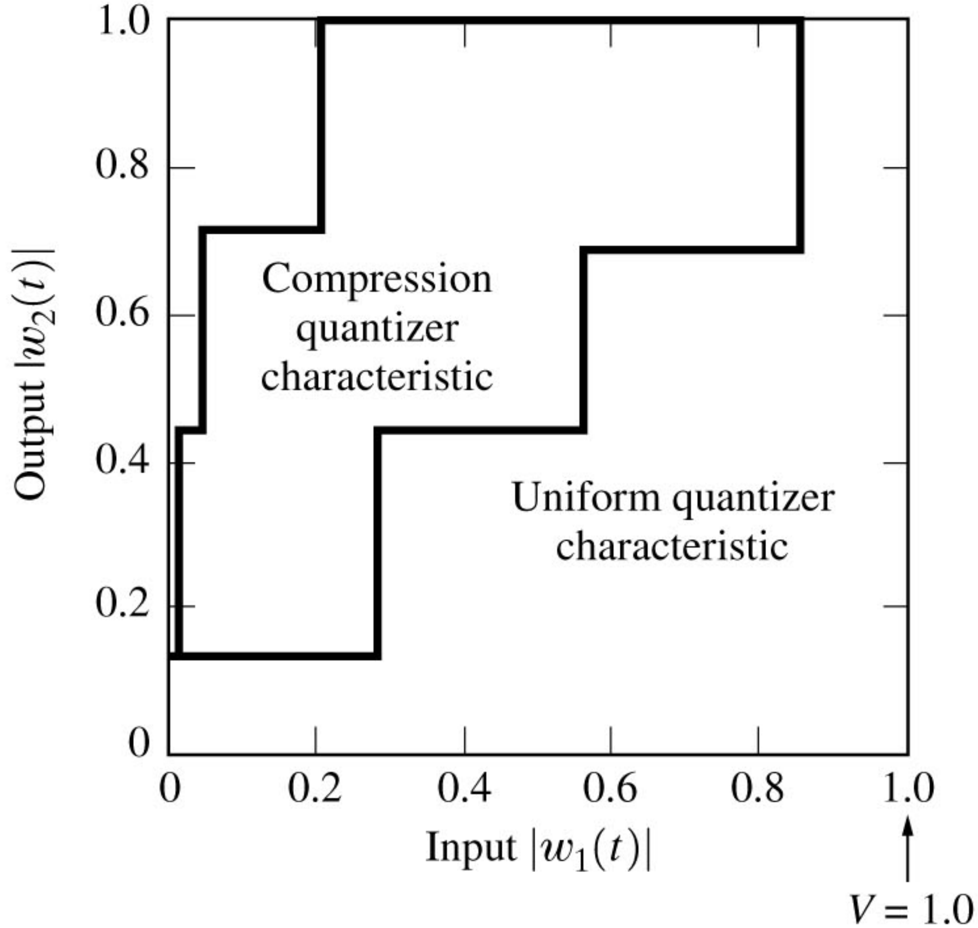
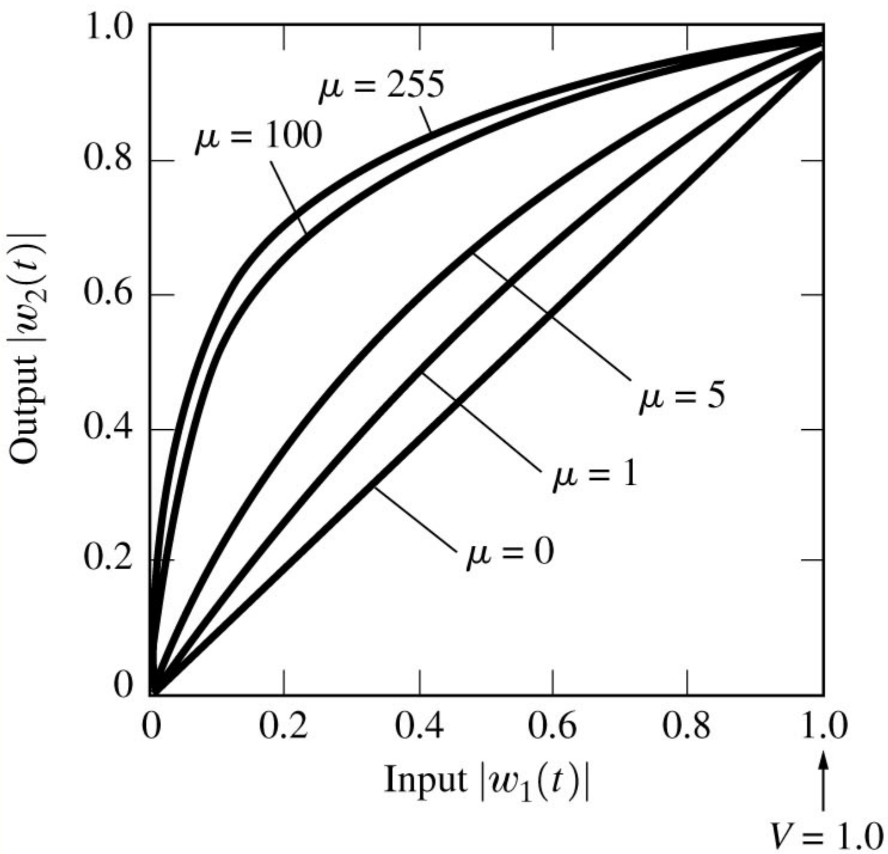


Figure 3–9 Compression characteristics (first quadrant shown).

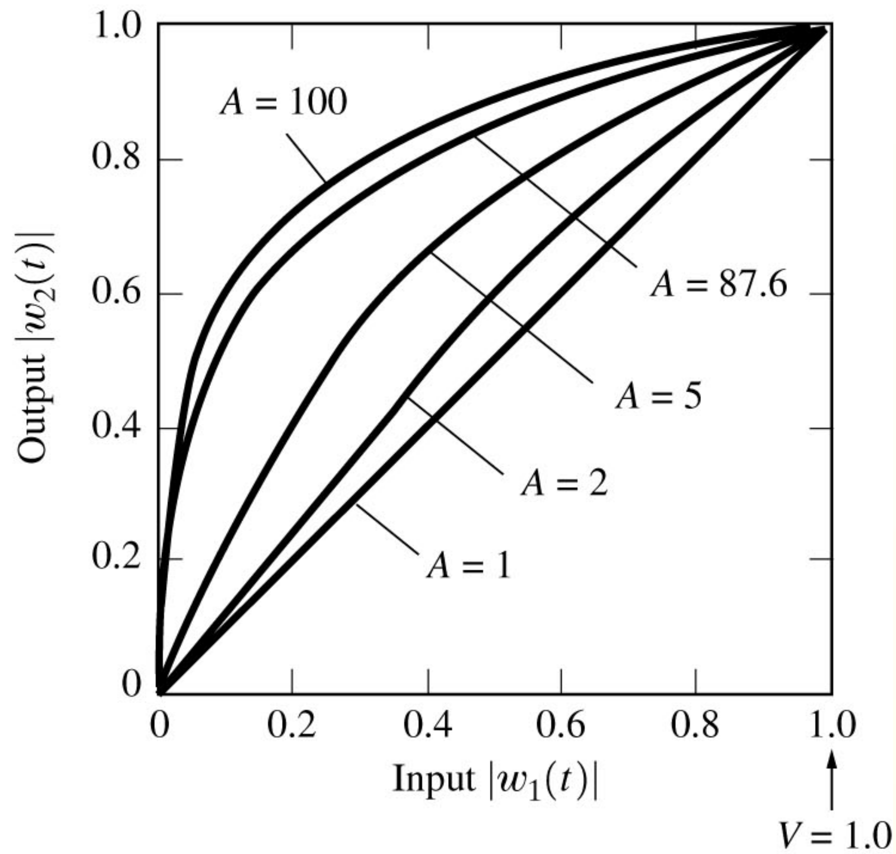


(a) $M = 8$ Quantizer Characteristic

Figure 3–9 Compression characteristics (first quadrant shown).

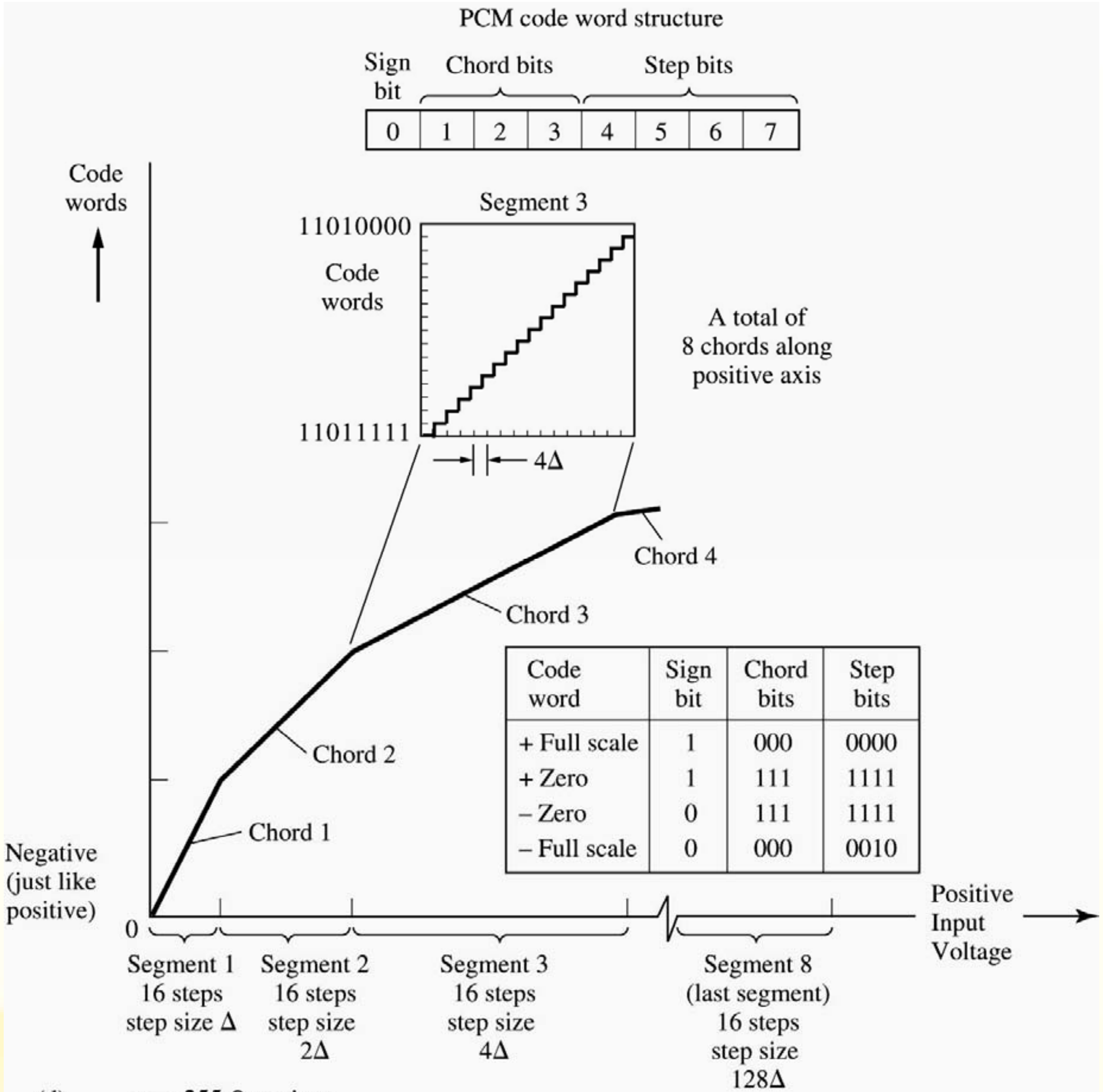


(b) μ -law Characteristic



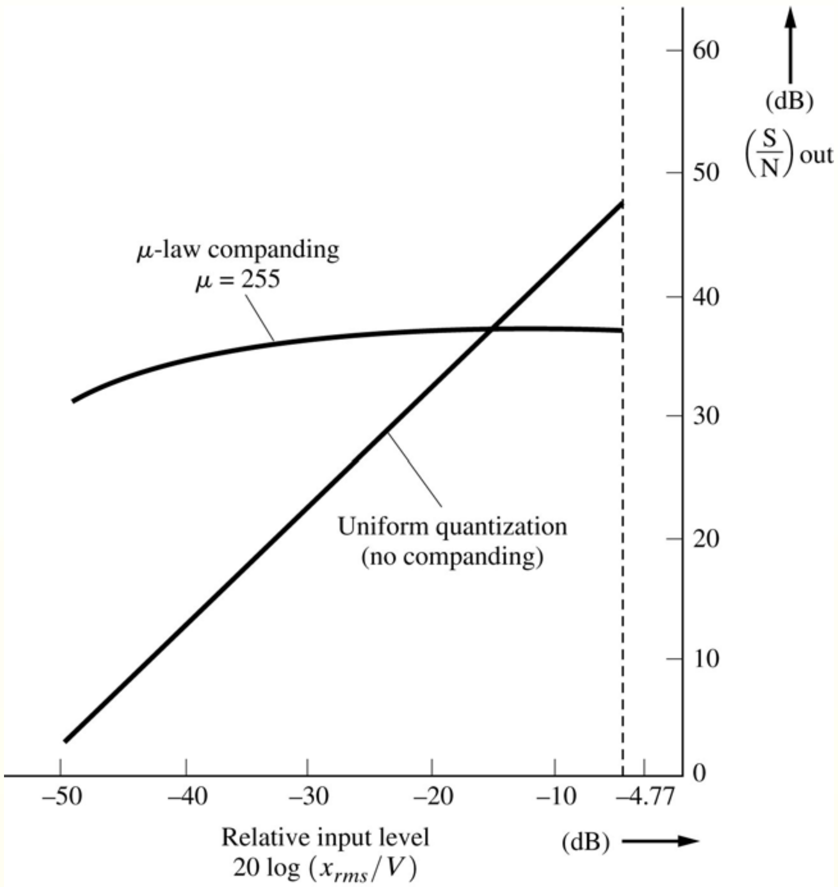
(c) A-law Characteristic

Figure 3–9 Continued



(d) $\mu = 255$ Quantizer

Figure 3–10 Output SNR of 8-bit PCM systems with and without companding.

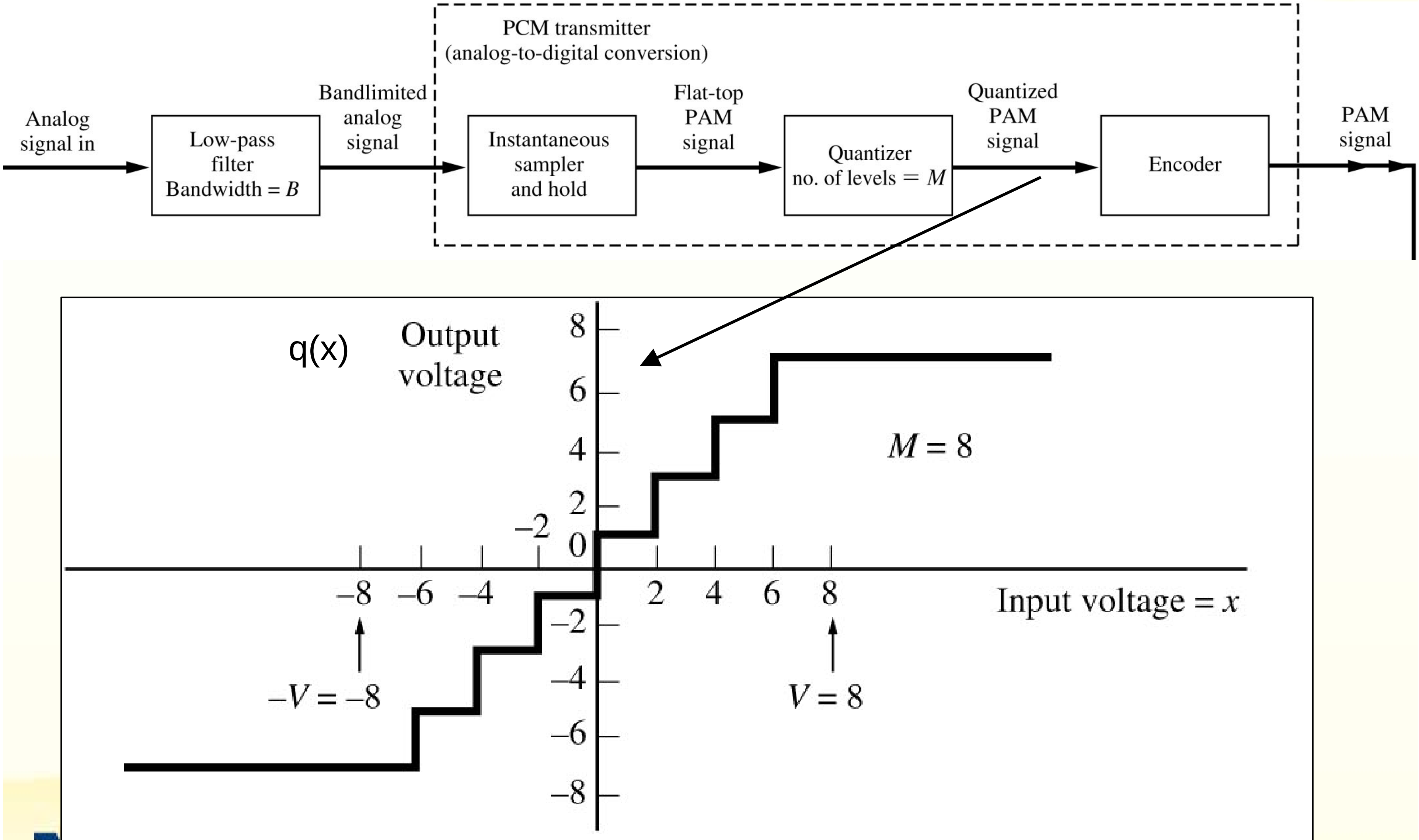


Quantization Noise, Analog to Digital Converter-A/D

EELE445

Lecture 13

Figure 3–7 PCM trasmission system.



Quantization

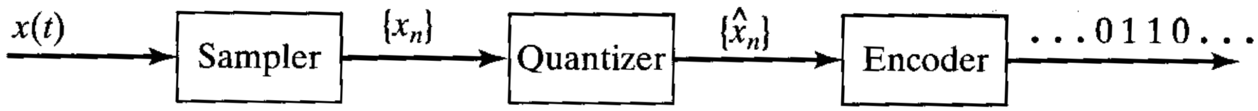
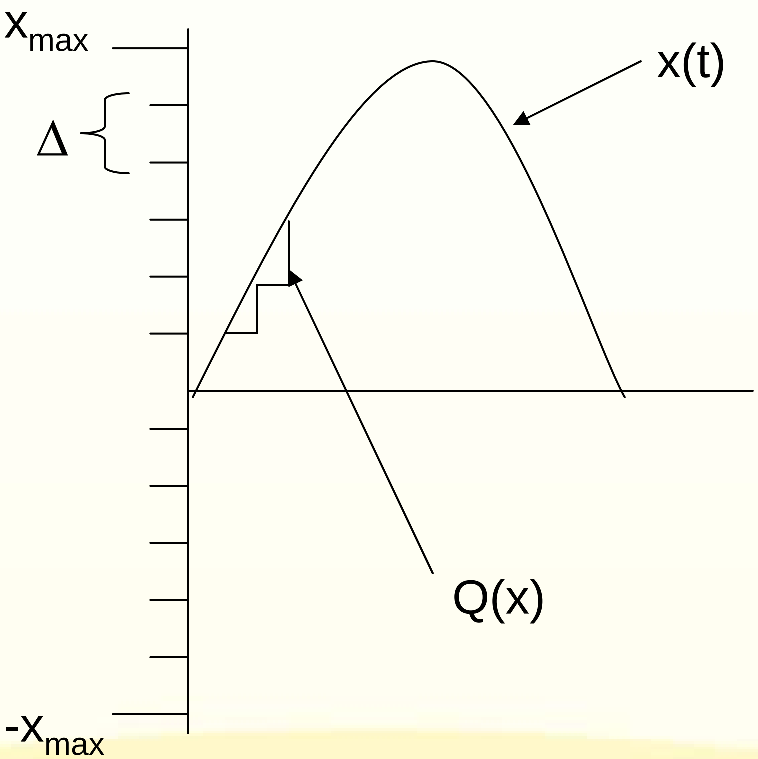
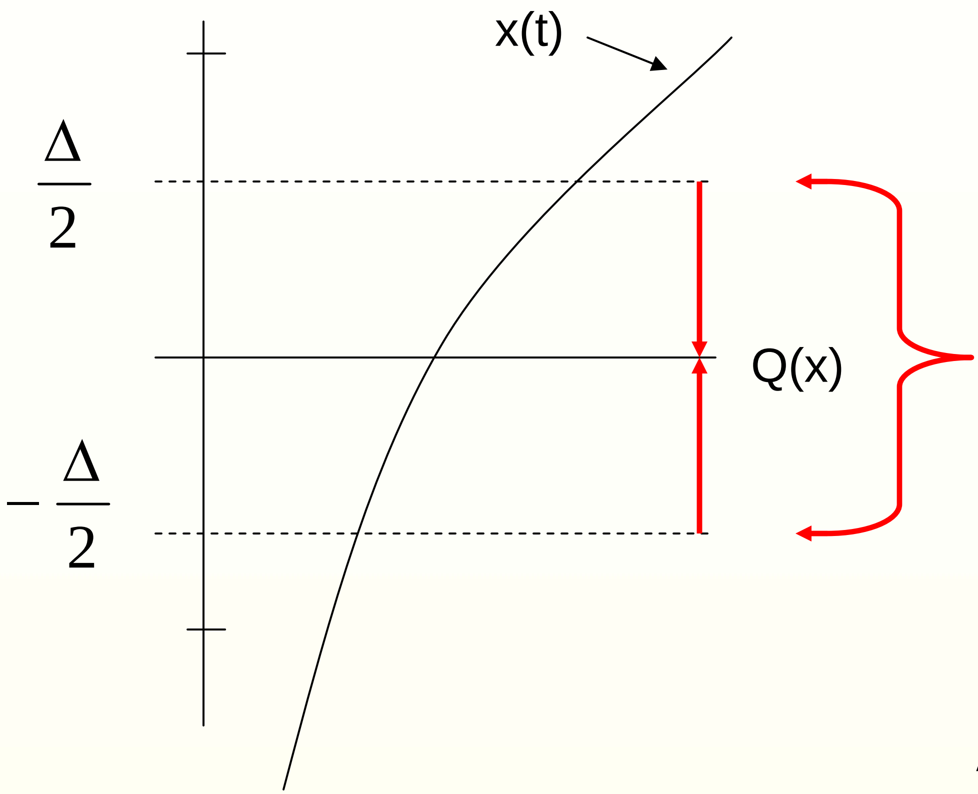


Figure 7.7 Block diagram of a PCM system.



$$\Delta = \frac{2x_{max}}{N} = \frac{x_{max}}{2^{n-1}}$$

Quantization – Results in a Loss of Information



$$\Delta = \frac{2x_{\max}}{N} = \frac{x_{\max}}{2^{n-1}}$$

After sampling, $x(t)=x_i$

$$x_i \in \mathcal{R}$$

After Quantization:

$$\mathrm{Q(x)} = \hat{\mathrm{x}}, \quad \mathrm{x} \in \mathcal{R}$$

Quantization Noise

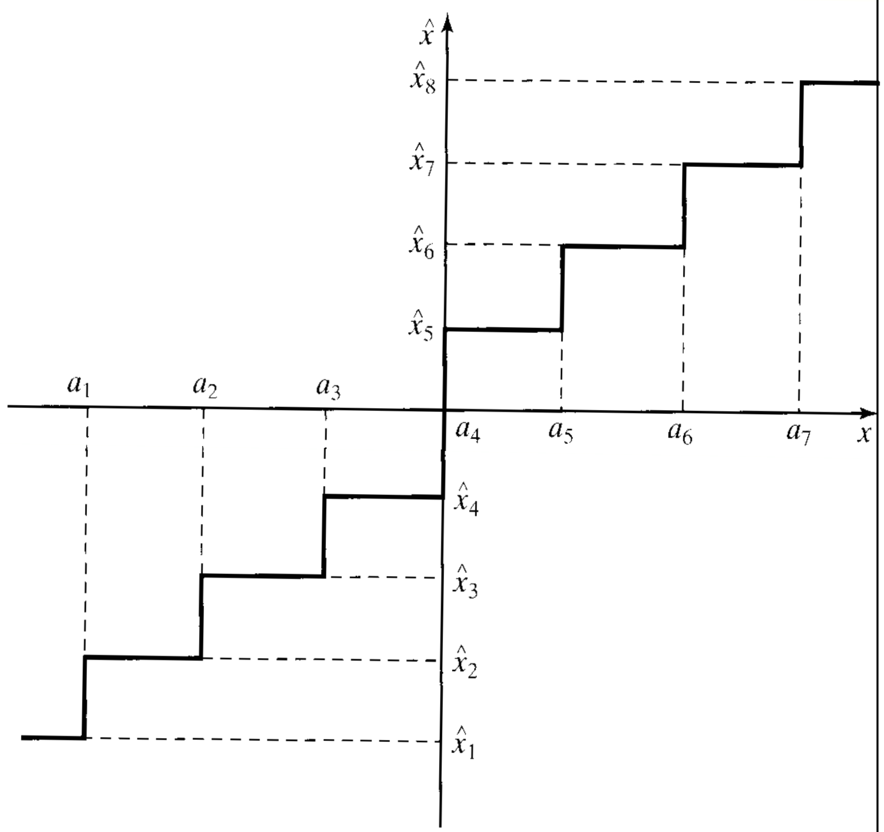


Figure 7.3 Example of an 8-level quantization scheme.

Quantization function:

$$Q(x) = \hat{x}_i \text{ for all } x \in \mathcal{R}_i .$$

Define the mean square <u>distortion</u>:

$$q(x) = (x - Q(x))^2 = \tilde{x}^2$$

$$and$$

$$|x - Q(x)| \le \frac{\Delta}{2}$$

Quantization

$$\langle q^2 \rangle = \frac{1}{\Delta}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} q^2 dq$$

$$= \frac{\Delta^2}{12}$$

$$= \boxed{\frac{(x_{\max})^2}{3N^2} = P_{nq} \quad \textit{the quantization noise}}$$

where $N=2^n$ and x_{max} is ½ the A/D input range

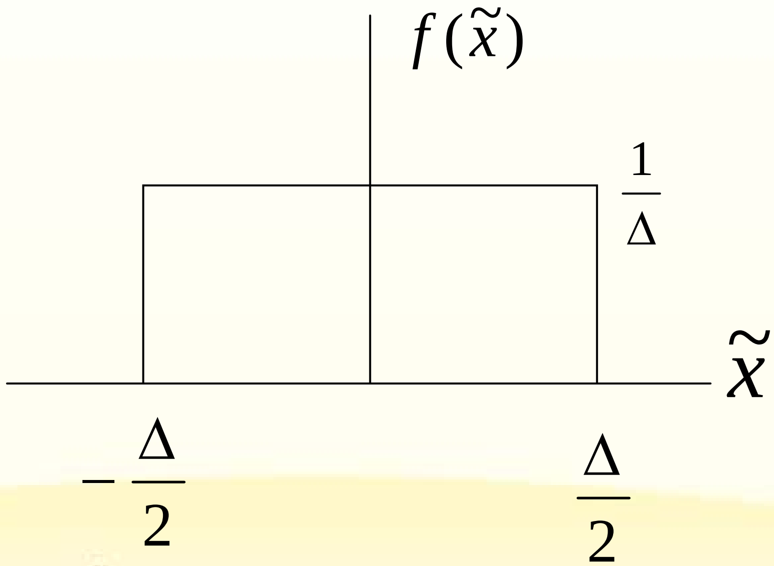
Quantization Noise

Since X is a random variable, so are $\hat{X}$ and $\tilde{X}$:

So we can define the mean squared error distortion as:

$$D = E[d(X, \hat{X})] = E[(X - Q(X))^2].$$

The pdf of the error is uniformly distributed $\tilde{X} = X - Q(X)$



$$f(\tilde{x}) = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} \le \tilde{x} \le \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

SQNR – Signal to Quantization Noise Ratio

Definition ' If the random variable X is quantized to $Q(X)$, the *signal-to-quantization noise ratio* (SQNR) is defined by

$$\text{SQNR} = \frac{E[X^2]}{E[X - Q(X)]^2}.$$

When dealing with signals, the quantization noise power is

$$P_{\tilde{X}} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} E\left[X(t) - Q(X(t))\right]^2 dt$$

and the signal power is

$$P_X = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} E[X^2(t)]dt.$$

Hence, the signal-to-quantization noise ratio is

$$\text{SQNR} = \frac{P_X}{P_{\tilde{X}}}.$$

SQNR – Signal to Quantization Noise Ratio

Example 7.2.2

Determine the SQNR for the quantization scheme given in Example 7.2.1.

Solution From Example 7.2.1, we have $P_X = 400$ and $P_{\tilde{X}} = D = 33.38$. Therefore,

$$\text{SQNR} = \frac{P_X}{P_{\tilde{X}}} = \frac{400}{33.38} = 11.98 \approx 10.78 \quad \text{dB.} \qquad \blacksquare$$

Example of SQNR for full scale sinewave done on board

SQNR – Signal to Quantization Noise Ratio

P_x may be found using:

$$P_X = R_X(\tau)|_{\tau=0}$$

$$= \int_{-\infty}^{\infty} S_X(f)\,df$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx.$$

SQNR – Signal to Quantization Noise Ratio

The distortion, or "noise", is therefore:

$$E[\tilde{X}^2] = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \frac{1}{\Delta} \tilde{x}^2 d\tilde{x} = \frac{\Delta^2}{12} = \frac{x_{\max}^2}{3N^2} = \frac{x_{\max}^2}{3 \times 4^{\nu}}$$

$$\mathrm{SQNR} = \frac{P_X}{\overline{\tilde{X}^2}} = \frac{3 \times N^2 P_X}{x_{\max}^2} = \frac{3 \times 4^{\nu} P_X}{x_{\max}^2}$$

Where P_X is the power of the input signal

SQNR – Linear Quantization

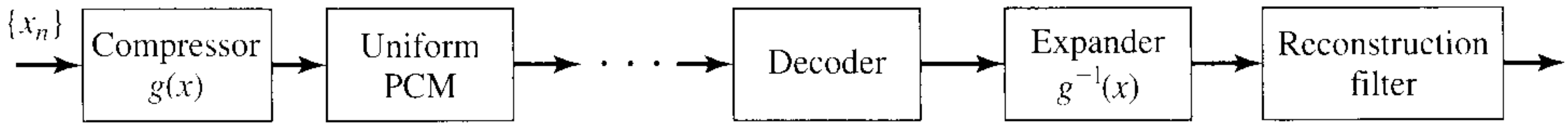
$$E\left[X^2\right] \le x_{\max}^2$$

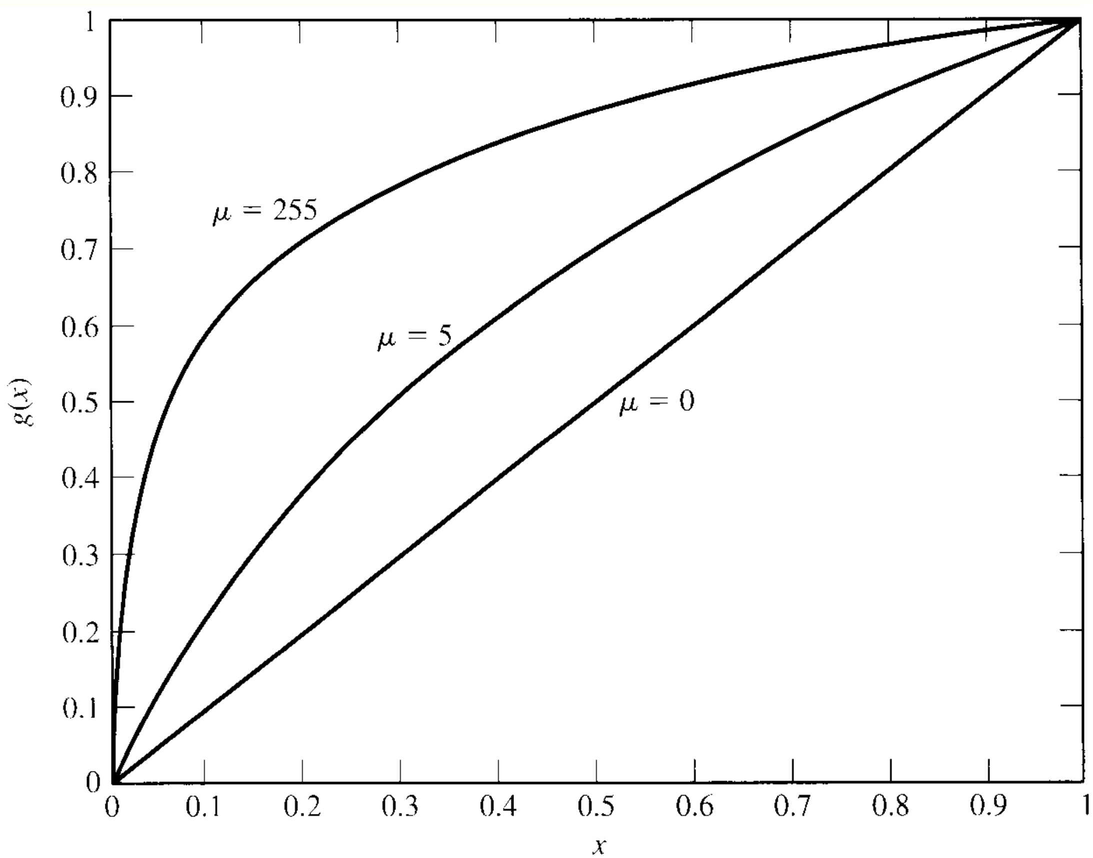
$$\frac{P_x}{x_{\max}^2} < 1$$

The SQNR decreases as
The input dynamic range
increases

$$\left.\mathrm{SQNR}\right|_{\mathrm{dB}} \approx 10\log_{10}\frac{P_X}{x_{\max}^2} + 6\nu + 4.8.$$

μ-Law Nonuniform PCM
used to increase SQNR for given P_x , xmax, and n

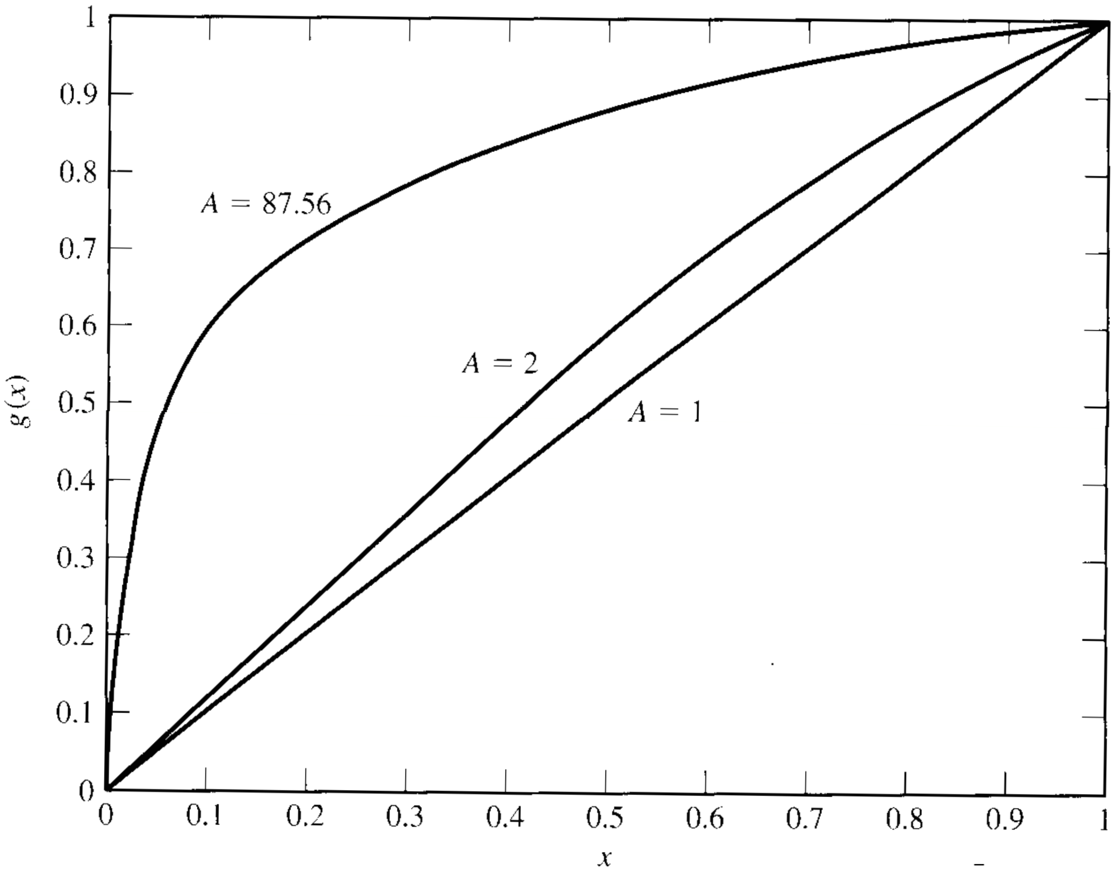




$$g(x) = \frac{\log(1 + \mu|x|)}{\log(1 + \mu)} \operatorname{sgn}(x).$$

μ=255 U.S

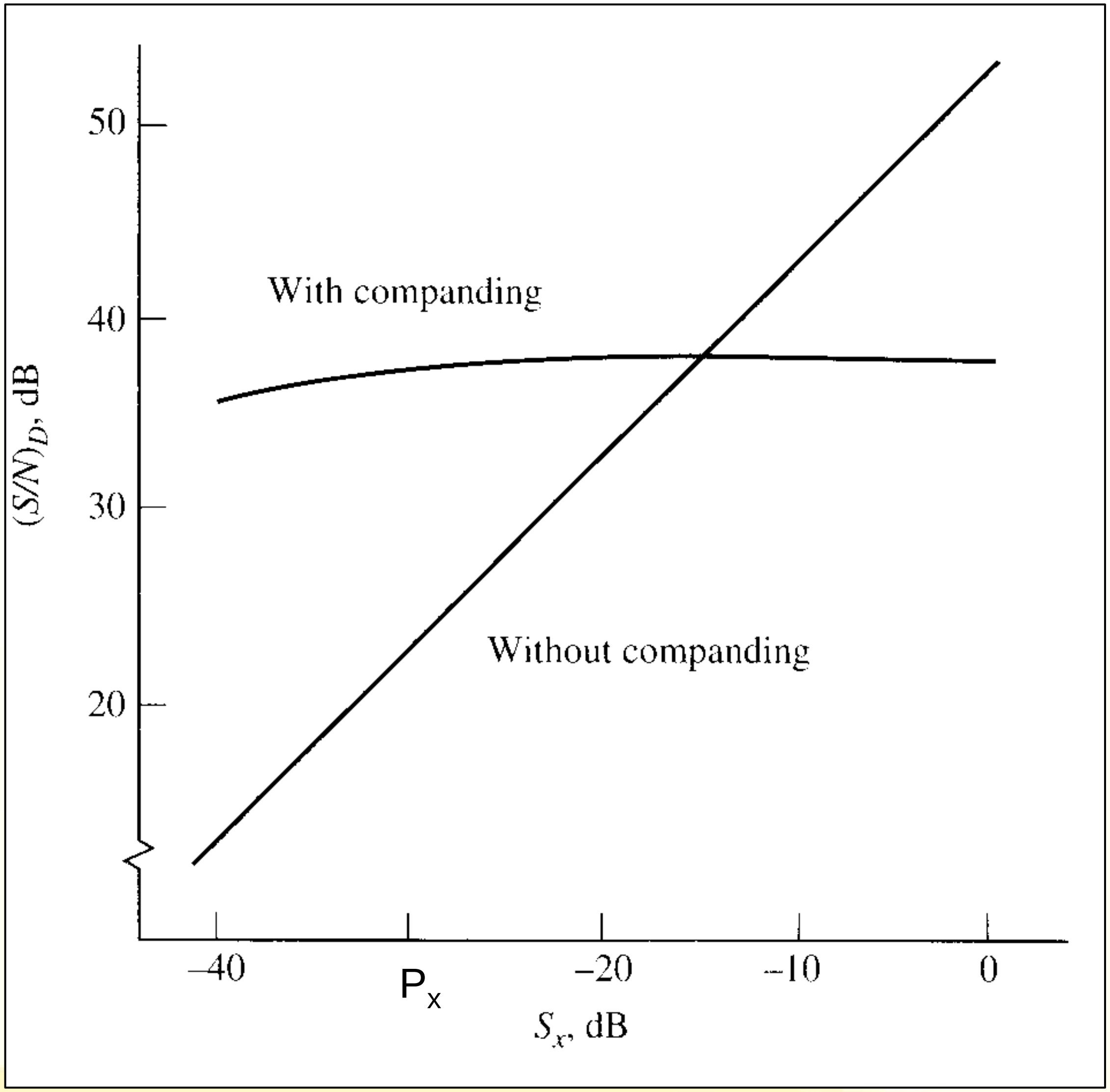
a-Law Nonuniform PCM



a=87.56 U.S

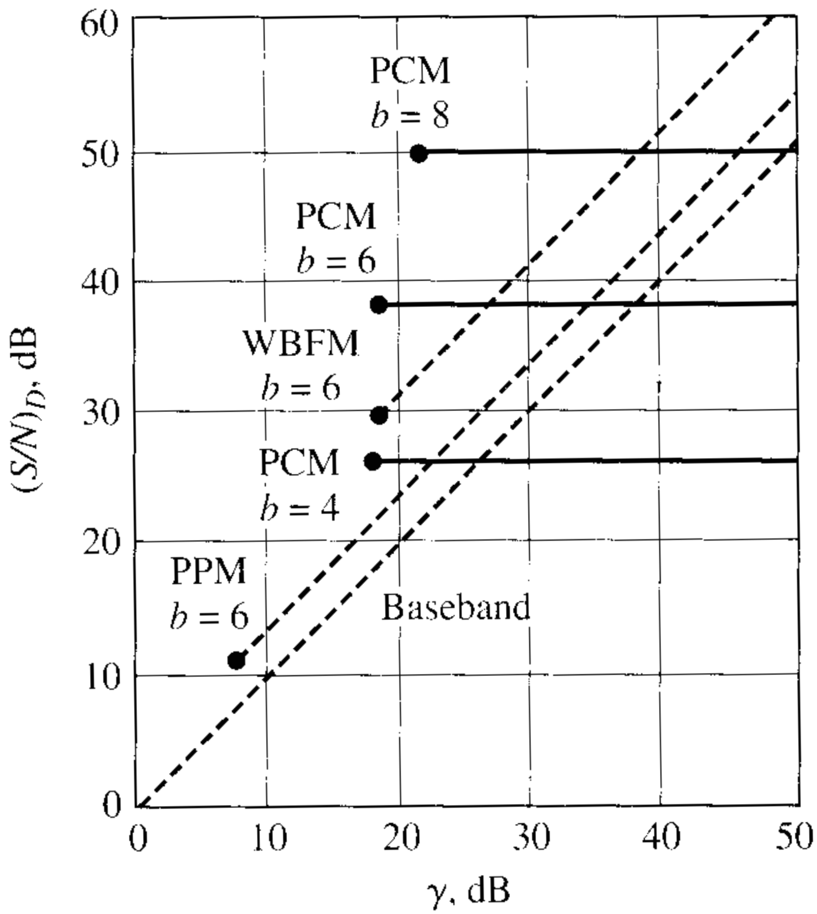
$$g(x) = \frac{1 + \log A|x|}{1 + \log A}\,\mathrm{sgn}(x),$$

μ-Law v.s. Linear Quantization



8 bit
P_x is signal power
Relative to full scale

Pulse Code Modulation, PCM, Advantage
alog systems



Performance comparison of PCM and analog modulation.

b is the number of bits

γ is $(S/N)_{baseband}$
Relative to full scale

PPM is pulse position
modulation