

EELE445-14

Lecture 3,4
Power, Energy,
Time average operator
section 2.1



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Lecture 3,4 EE445 - Outcomes

- In this lecture you:
 - be able to use the time average operator $\langle \cdot \rangle$ for finite time duration signals and periodic signals
 - Be able to find the rms value of a waveform
 - be able to find the power or energy of a waveform in the time domain
 - Parseval's Theorem, find the power or energy of a waveform in the frequency domain
 - Concept of PSD and ESD



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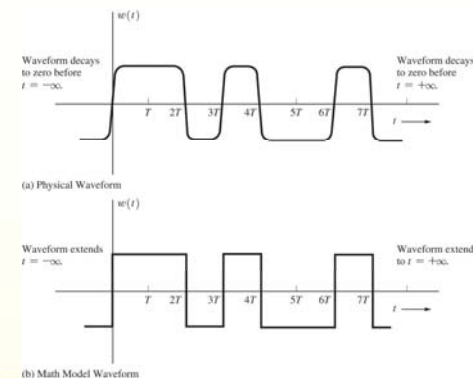
Physically Realizable Waveforms

- measurable in the laboratory
1. waveform has significant nonzero values over a composite time interval for a finite time duration
 2. continuous function of time
 3. finite peak amplitude
 4. real function, no complex values



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Figure 2-1 Physical and mathematical waveforms.



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Time average operator

Time Average Operator (mean, of $w(t)$):

$$\langle w(t) \rangle = W_{dc} = \bar{w} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} w(t) dt$$



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Time average operator

Time Average Operator for signal of finite time duration:

$$\langle w(t) \rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} w(t) dt \quad \langle w_n \rangle = \frac{1}{N} \sum_{n=1}^N w_n$$

For a Periodic Signal:

$$w(t) = w(t + aT_0) \quad \text{for any real } a$$

$$\langle w(t) \rangle = \frac{1}{T_0} \int_{-\frac{T_0}{2} + a}^{\frac{T_0}{2} + a} w(t) dt \quad \text{for any real } a$$



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Root Mean Square- rms

$$W_{\text{meansquare}} = \langle w^2(t) \rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} w^2(t) dt \quad (\text{Power } R = 1)$$

RMS = "root mean square" of a signal of duration T:

$$W_{rms} = \sqrt{\langle w^2(t) \rangle} = \sqrt{\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} w^2(t) dt}$$



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RMS

For periodic waveforms :

$$\sqrt{\langle w^2(t) \rangle} = \sqrt{\frac{1}{T_0} \int_{\frac{T_0}{2} - \tau}^{\frac{T_0}{2} + \tau} w^2(t) dt} = \text{rms value of } w(t)$$

- where τ is an arbitrary time shift.
- Choose τ to simplify the integrals!



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RMS example

RMS of a sinwave:

$$\sqrt{\langle [A \cos(\omega_0 t)]^2 \rangle} = \frac{A}{\sqrt{2}} \quad \text{appendix A-4}$$

use: $2 \cos^2(x) = 1 + \cos(2x)$

RMS of a SquareWave = A

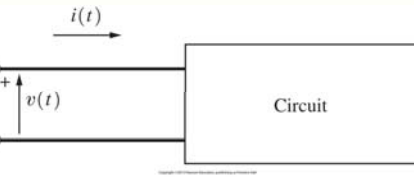


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Example 2-2 EVALUATION OF POWER

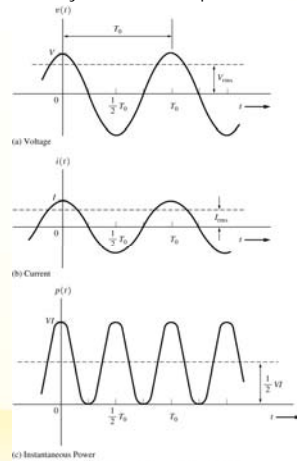
Let the circuit of Fig. 2-2 contain a 120-V, 60-Hz fluorescent lamp wired in a high-power-factor configuration. Assume that the voltage and current are both sinusoids and in phase (unity power factor), as shown in Fig. 2-3.[†] The DC value of this (periodic) voltage waveform is

[†] Two-lamp fluorescent circuits can be realized with a high-power-factor ballast that gives an overall power factor greater than 90% [Fink and Beaty, 1978].



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Figure 2-3 Steady-state waveshapes for Example 2-2.



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Example 2-2 EVALUATION OF POWER

$$V_{dc} = \langle v(t) \rangle = \langle V \cos \omega_0 t \rangle$$

$$= \frac{1}{T_0} \int_{T_0/2}^{T_0/2} V \cos \omega_0 t dt = 0 \quad (2-8)$$

where $\omega_0 = 2\pi/T_0$ and $f_0 = 1/T_0 = 60$ HZ. Similarly, $I_{dc} = 0$. The instantaneous power is

$$p(t) = (V \cos \omega_0 t)(I \cos \omega_0 t) = \frac{1}{2} VI (1 + \cos 2\omega_0 t) \quad (2-9)$$

The average power is

$$P = \left\langle \frac{1}{2} VI (1 + \cos 2\omega_0 t) \right\rangle$$

$$= \frac{VI}{2T_0} \int_{T_0/2}^{T_0/2} (1 + \cos 2\omega_0 t) dt$$

$$= \frac{VI}{2} \quad (2-10)$$

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Power of a signal

instantaneous Power $p(t) = v(t)i(t)$

$$\text{average Power } \langle p(t) \rangle = \langle v(t)i(t) \rangle = \text{Real} \left\{ \left\langle \frac{v^2(t)}{Z_{\text{load}}} \right\rangle \right\}$$

$$= \frac{1}{R} \langle v^2(t) \rangle \text{ when } Z = R + j0$$

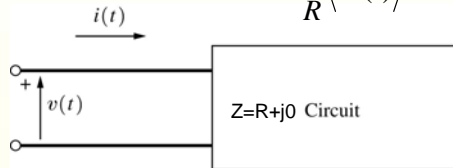


Figure 2-2 Polarity convention used for voltage and current.

Power of a signal

Average Power in signal of duration T :

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)i(t) dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} p(t) dt$$

$p(t)$ is the instantaneous power, P is the average power over time T

$$P = \frac{E}{T} = \frac{1}{R} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v^2(t) dt \equiv \frac{v_{\text{rms}}^2}{R}$$

For Normalized Power:
 $R=1$ ohm

$$P = \frac{R}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} i^2(t) dt \equiv i_{\text{rms}}^2 R$$

Energy of a Signal

Energy in signal of duration T :

$$P = \frac{E}{T} = \frac{1}{RT} \int_{-\frac{T}{2}}^{\frac{T}{2}} v^2(t) dt \quad E = PT$$

$$E = \frac{1}{R} \int_{-\frac{T}{2}}^{\frac{T}{2}} v^2(t) dt = R \int_{-\frac{T}{2}}^{\frac{T}{2}} i^2(t) dt$$

We typically calculate normalized Energy or Power, $R = 1$

A communication metric: Signal to Noise Ratio in dB

$$(S/N)_{\text{dB}} \equiv 10 \text{Log} \left(\frac{P_{\text{signal}}}{P_{\text{noise}}} \right) = 10 \text{Log} \left(\frac{\langle s(t)^2 \rangle}{\langle n(t)^2 \rangle} \right)$$

$$= 20 \text{Log} \left(\frac{V_{\text{rms signal}}}{V_{\text{rms noise}}} \right)$$

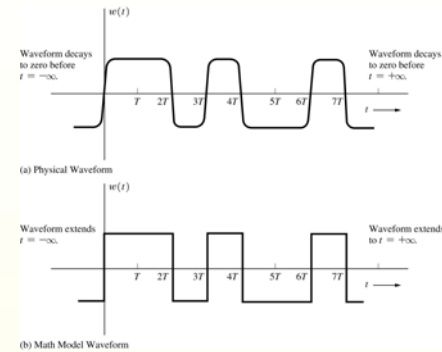
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Lecture 4 The Fourier Transform and the Frequency Domain



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Figure 2–1 Physical and mathematical waveforms.



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The Fourier Transform

We use the Fourier transform to determine the frequencies of the sinusoids present in a non-repetitive waveform.

The Fourier series may be considered the Fourier transform of a non-repetitive waveform that is repeated continuously.



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Fourier Transform

$$W(f) = \mathcal{F}[w(t)] = \int_{-\infty}^{\infty} w(t) \cdot e^{-j2\pi ft} dt$$

What is :

$$\int x(t)y(t)dt ???$$

Correlation coeff

$$C = \frac{\int x(t)y(t)dt}{\sqrt{\int x(t)^2 dt \cdot \int y(t)^2 dt}}$$

$$-1 < C < 1$$



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Fourier Transform

$$W(f) = \mathcal{F}[w(t)] = \int_{-\infty}^{\infty} [w(t)] \bullet e^{-j2\pi ft} dt$$

So the Fourier transform is answering the question:

how much of $e^{-j2\pi ft}$ is in $w(t)$?

$$e^{-jx} = \cos(x) - j \sin(x)$$

Figure 2-4 Spectrum of a sine wave.

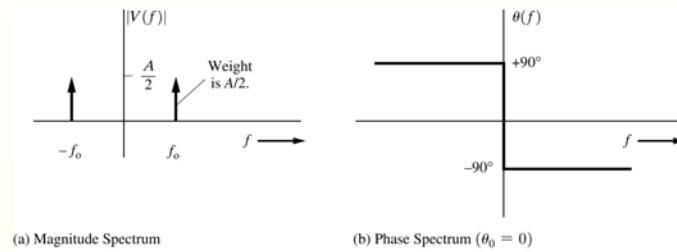


Figure 2-5 Waveshapes and corresponding symbolic notation.

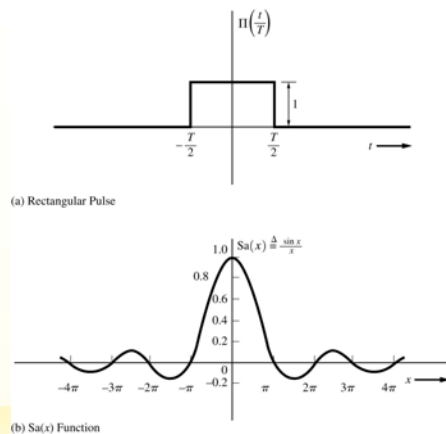


Figure 2-5 Waveshapes and corresponding symbolic notation.

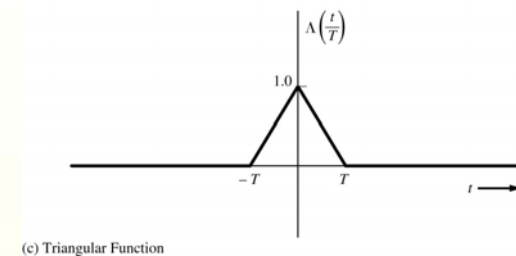


Figure 2-6 Spectra of rectangular, $(\sin x)/x$, and triangular pulses.

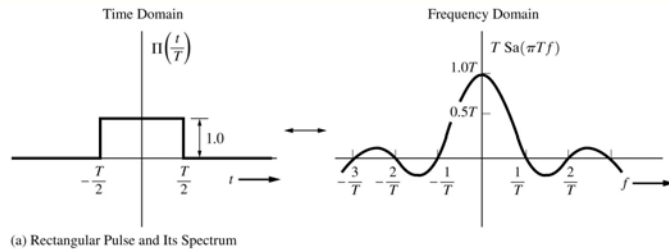
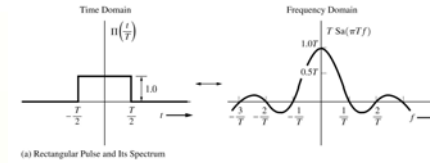


Figure 2-6 Spectra of rectangular, $(\sin x)/x$, and triangular pulses.



Time - bandwidth product or bandwidth - duration product :

$$T_{rms} B_{rms} \leq \frac{1}{4\pi}$$

the Gaussian pulse, $w(t) = e^{-\pi t^2}$ satisfies the condition with the equality sign.

Figure 2-6 Spectra of rectangular, $(\sin x)/x$, and triangular pulses.

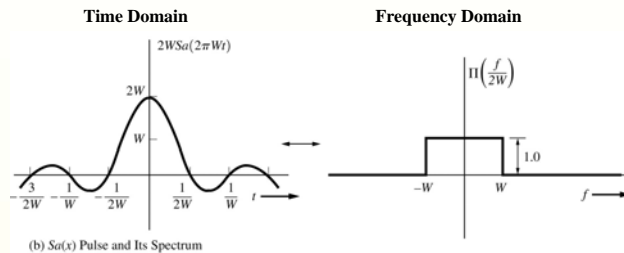
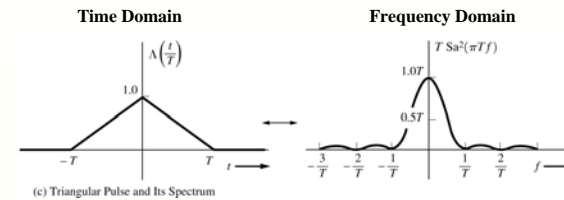


Figure 2-6 Spectra of rectangular, $(\sin x)/x$, and triangular pulses.



Fourier Transform Theorems

TABLE 2-1 SOME FOURIER TRANSFORM THEOREMS^a

Operation	Function	Fourier Transform
Linearity	$a_1 w_1(t) + a_2 w_2(t)$	$a_1 W_1(f) + a_2 W_2(f)$
Time delay	$w(t - T_d)$	$W(f) e^{-j\omega T_d}$
Scale change	$w(at)$	$\frac{1}{ a } W\left(\frac{f}{a}\right)$
Conjugation	$w^*(t)$	$W^*(-f)$
Duality	$W(t)$	$w(-f)$
Real signal frequency translation [$w(t)$ is real]	$w(t) \cos(\omega_c t + \theta)$	$\frac{1}{2}[e^{j\theta} W(f - f_c) + e^{-j\theta} W(f + f_c)]$
Complex signal frequency translation	$w(t) e^{j\omega_c t}$	$W(f - f_c)$

These theorems are useful for the homework and the exams!



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Fourier Transform Theorems

Bandpass signal	$\text{Re}\{g(t) e^{j\omega_c t}\}$	$\frac{1}{2}[G(f - f_c) + G^*(-f - f_c)]$
Differentiation	$\frac{d^n w(t)}{dt^n}$	$(j2\pi f)^n W(f)$
Integration	$\int_{-\infty}^t w(\lambda) d\lambda$	$(j2\pi f)^{-1} W(f) + \frac{1}{2} W(0) \delta(f)$
Convolution	$w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(\lambda) \cdot w_2(t - \lambda) d\lambda$	$W_1(f) W_2(f)$
Multiplication ^b	$w_1(t) w_2(t)$	$W_1(f) * W_2(f) = \int_{-\infty}^{\infty} W_1(\lambda) W_2(f - \lambda) d\lambda$
Multiplication by t^n	$t^n w(t)$	$(-j2\pi f)^{-n} \frac{d^n W(f)}{df^n}$

^a $\omega_c = 2\pi f_c$.

^b * denotes convolution as described in detail by Eq. (2-62).



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Fourier Transform Pairs

TABLE 2-2 SOME FOURIER TRANSFORM PAIRS

Function	Time Waveform $w(t)$	Spectrum $W(f)$
Rectangular	$\Pi\left(\frac{t}{T}\right)$	$T[\text{Sa}(\pi f T)]$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T[\text{Sa}(\pi f T)]^2$
Unit step	$u(t) \triangleq \begin{cases} +1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{j2\pi f} + \frac{1}{2} \delta(f)$
Signum	$\text{sgn}(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$

The rectangular function is used to mathematically truncate an infinite time waveform $w(t)$. It is the time window that $w(t)$ is analyzed over. What is the effect on the measurement of $w(t)$ as T approaches infinity? How about as T approaches 0?



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Fourier Transform Pairs

Impulse at $t = t_0$	$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Sinc	$\text{Sa}(2\pi W t)$	$\frac{1}{2W} \Pi\left(\frac{f}{2W}\right)$
Phasor	$e^{j(\omega t + \varphi)}$	$e^{j\varphi} \delta(f - f_0)$
Sinusoid	$\cos(\omega_c t + \varphi)$	$\frac{1}{2} e^{j\varphi} \delta(f - f_c) + \frac{1}{2} e^{-j\varphi} \delta(f + f_c)$
Gaussian	$e^{-\pi(t/t_0)^2}$	$t_0 e^{-\pi(f/f_0)^2}$
Exponential, one-sided	$\begin{cases} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{T}{1 + j2\pi f T}$
Exponential, two-sided	$e^{- t /T}$	$\frac{2T}{1 + (2\pi f T)^2}$
Impulse train	$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$f_0 \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$, where $f_0 = 1/T$

compare the Phasor and the sinusoid transforms.



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Energy and Power in a Waveform

Parseval's Theorem and Energy Spectral Density

Parseval's Theorem.

$$\int_{-\infty}^{\infty} w_1(t)w_2^*(t) dt = \int_{-\infty}^{\infty} W_1(f)W_2^*(f) df \quad (2-40)$$

If $w_1(t) = w_2(t) = w(t)$, then the theorem reduces to

Rayleigh's energy theorem, which is

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt = \int_{-\infty}^{\infty} |W(f)|^2 df \quad (2-41)$$

DEFINITION. The *energy spectral density (ESD)* is defined for energy waveforms by

$$\mathcal{E}(f) = |W(f)|^2 \quad (2-42)$$

where $w(t) \leftrightarrow W(f)$. $\mathcal{E}(f)$ has units of joules per hertz.

Using Eq. (2-41), we see that the total normalized energy is given by the area under the ESD function:

$$E = \int_{-\infty}^{\infty} \mathcal{E}(f) df \quad (2-43)$$



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Energy and Power in a Waveform

• Energy Spectral Density, ESD

$$E(f) \equiv |W(f)|^2 \quad W(f) \equiv \int_{-\infty}^{\infty} w(t) \bullet e^{-j2\pi ft} dt$$

• Power Spectral Density, PSD, for a truncated waveform of duration T

$$w_T(t) = \begin{cases} w(t), & -T/2 < t < T/2 \\ 0, & \text{elsewhere} \end{cases} = w(t) \Pi\left(\frac{t}{T}\right) \quad (2-64)$$

Using Eq. (2-13), we obtain the average normalized power:

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} w^2(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} w_T^2(t) dt$$

By the use of Parseval's theorem, Eq. (2-41), the average normalized power becomes

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} |W_T(f)|^2 df = \int_{-\infty}^{\infty} \left(\lim_{T \rightarrow \infty} \frac{|W_T(f)|^2}{T} \right) df \quad (2-65)$$



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Energy and Power in a Waveform

DEFINITION. The *power spectral density (PSD)* for a deterministic power waveform is[†]

$$\mathcal{P}_w(f) \triangleq \lim_{T \rightarrow \infty} \left(\frac{|W_T(f)|^2}{T} \right) \quad (2-66)$$

where $w_T(t) \leftrightarrow W_T(f)$ and $\mathcal{P}_w(f)$ has units of watts per hertz.

Note that the PSD is always a real nonnegative function of frequency. In addition, the PSD is not sensitive to the phase spectrum of $w(t)$, because that is lost due to the absolute value operation used in Eq. (2-66). From Eq. (2-65), the normalized average power is[†]

$$P = \langle w^2(t) \rangle = \int_{-\infty}^{\infty} \mathcal{P}_w(f) df \quad (2-67)$$

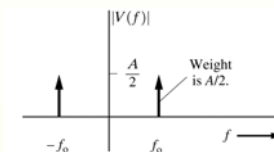
That is, the area under the PSD function is the normalized average power.



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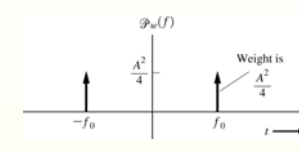
Energy and Power in a Waveform

Figure 2-4 Spectrum of a sine wave.



(a) Magnitude Spectrum

Figure 2-9 Power spectrum of a sinusoid.



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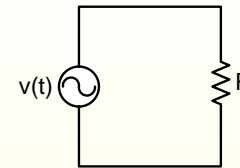
LECTURE 5 Examples, Fourier Series from the Fourier Transform



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Example- Power and Energy in Time.

A 1 KHz sinewave source is applied to a 50 ohm resistor.
find the power delivered to the load.



$$\begin{aligned} R &= 50 \, \Omega \\ v(t) &= A \cos(2\pi f t) \\ A &= 1 \, \text{volt} \\ f &= 1000 \, \text{Hz} \\ T_o &= 10^{-3} \, \text{sec} \end{aligned}$$



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Example- Power and Energy in Time.

For a periodic Signal :

$$\begin{aligned} P &= \frac{1}{R} \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} v^2(t) dt \\ &= \frac{A^2}{2T_o} \left[\frac{T_o}{2} - -\frac{T_o}{2} \right] \\ &= \frac{A^2}{2} \text{ Mean Squared volts (volts}^2\text{)} \\ \text{so} \\ P &= \frac{A^2}{2R} = \frac{1^2}{2 \bullet 50} = 0.01 \, \text{Watt} = 10 \, \text{dBm} \end{aligned}$$

$$\begin{aligned} \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} v^2(t) dt &= \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} (A \cos(2\pi f t))^2 dt \\ &= \frac{A^2}{2T_o} \left[\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} 1 dt + \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} \cos(4\pi f t) dt \right] \end{aligned}$$

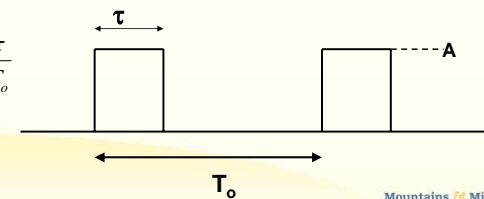


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Fourier Series from Fourier Transforms

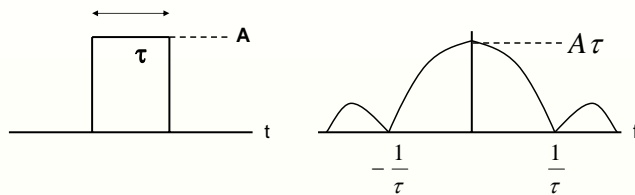
1. Find the truncated signal period $V_{T_o}(t)$
2. Determine the Fourier Transform of the waveform $V_{T_o}(t)$
3. Find the value of the Fourier Series coefficients by:
 1. Replacing f in the Transform with $f=n/T_o$
 2. Scaling the magnitude of the Transform by $1/T_o$

$$\text{Duty Cycle, } D = \frac{\tau}{T_o}$$



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Fourier Series from Fourier Transforms



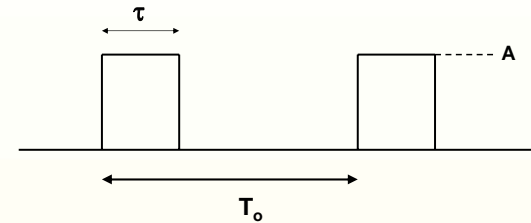
From Fourier Transform Table:

$$F\left(A \cdot \Pi\left[\frac{t}{\tau}\right]\right) = A\tau \text{Sa}(\pi f\tau)$$



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Fourier Series from Fourier Transforms

now make a repetitive pulse with period T_o 

Let $f = \frac{n}{T}$ and scale by $\frac{1}{T_o}$ $c_n = \frac{A\tau}{T_o} \text{Sa}\left[\frac{n\pi\tau}{T_o}\right] \therefore$



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Properties of a Repetitive Rectangular Pulse

- The null bandwidth, B_n of the spectrum is equal to the inverse of the pulse width, τ .

$$B_n = \frac{1}{\tau}$$

- The harmonic frequencies are n/T_o .

- The number of harmonics, n , from DC to the spectral null is: $n = \frac{1}{D} = \frac{T_o}{\tau}$

- Approximately 90 percent of the power of the signal is in the spectrum from DC to the null bandwidth frequency, only 10% of the signal power is in the spectrum frequency range from the null bandwidth frequency to infinity!



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Repetitive Rectangular Pulse

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***** Definitions *****

$$\delta(f, R) = \text{if}(f = R, 1, 0) \quad \text{sinc}(x) = \text{if}\left(x = 0, 1, \frac{\sin(\pi \cdot x)}{\pi \cdot x}\right)$$

$$\Lambda(t, T) = \text{if}\left(\left|t\right| \leq \frac{T}{2}, 1 - \left|\frac{2t}{T}\right|, 0\right) \quad \text{rect}(t, T) = \text{if}\left(\left|t\right| \leq \frac{T}{2}, 1, 0\right)$$

***** set up waveform *****

$$T := 1\text{-ms} \quad \text{period} \quad D := 20\% \quad \text{duty cycle} \quad \tau := D \cdot T = 2 \times 10^{-4} \text{ s} \quad \text{pulse width}$$

$$A := 2\text{V} \quad \frac{1}{T} = 1 \times 10^3 \frac{1}{\text{s}} \quad \text{pps} \quad \frac{1}{\tau} = 5 \times 10^3 \frac{1}{\text{s}} \quad \text{First null}$$

***** Set up Sweep Parameters *****



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Repetitive Rectangular Pulse

*****Pulse shape Pt Definition *****

- ☒ Assume a rectangular pulse shape with an amplitude of 1

$$X_T(f) := A \cdot \tau \cdot \text{sinc}(f \tau) \quad \text{Single Pulse Fourier Transform, "double sided"}$$

$$n := -25, -24 \dots 25$$

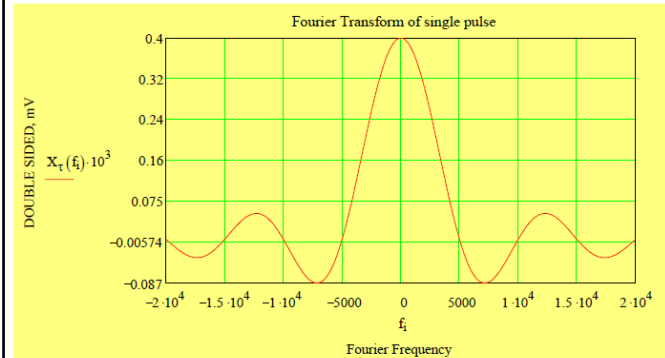
$$X_T(n) := \frac{A \cdot \tau}{T} \cdot \text{sinc}\left(\frac{n \cdot \tau}{T}\right) \quad \text{Fourier Series for repetitive pulse "double sided"}$$

$$X_T(0) = 0.4 \text{ V} \quad \text{DC value}$$



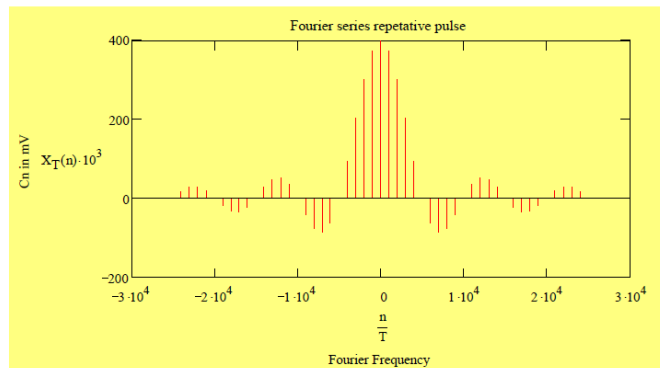
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Single Rectangular Pulse



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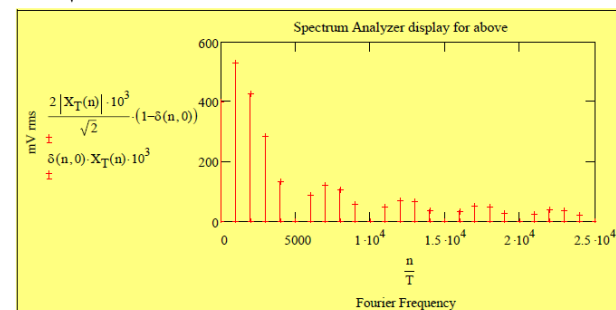
Repetitive Rectangular Pulse



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Repetitive Rectangular Pulse

$$\frac{2 |X_T(1)|}{\sqrt{2}} = 0.529 \text{ V} \quad \text{level of fundamental as seen by SA} \quad n := 0 \dots 25$$



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