FM - Frequency Modulation
PM - Phase Modulation

EELE445-14
Lecture 30

DSB-SC, AM, FM and PM

**DSB - SC Complex Envelope:** \[ g(t) = A_c m(t) \]

**AM Complex Envelope:** \[ g(t) = A_c (1 + m(t)) \]

**SSB - SC Complex Envelope:** \[ g(t) = A_c [m(t) \pm j\hat{m}(t)] \]

**PM Complex Envelope:** \[ g(t) = A_c e^{jD_p m(t)} \]

**FM Complex Envelope:** \[ g(t) = A_c e^{jD \int_{-\infty}^{\infty} m(\sigma) d\sigma} \]
FM and PM

\[ g(t) = R(t)e^{j\theta(t)} = A_c e^{j\theta(t)} \quad \text{Complex Envelope} \]

\[ R(t) = |g(t)| = A_c \quad \text{the real envelope is a constant} \]

\[ \rightarrow \text{power is constant} \]

Transmitted angle-modulated signal:

\[ s(t) = \text{Re}[g(t)e^{j\omega t}] = A_c \cos[\omega t + \theta(t)] \]

\[ \text{for PM: } \theta(t) = D_p m(t) \]

\[ D_p = \text{phase sensitivity or modulation constant } \frac{\text{rad}}{\text{volt}} \]

\[ \text{for FM: } \theta(t) = D_f \int_{-\infty}^{t} m(\sigma)d\sigma \]

\[ D_f = \text{frequency deviation or modulation constant } \frac{\text{rad}}{\text{volt} - \text{sec}} \]

\[ D_f = 2\pi \frac{\text{Hz}}{\text{volt}} \]
FM and PM

Relationship between $m_f(t)$ and $m_p(t)$:

$$m_f(t) = \frac{D_p}{D_f} \left[ \frac{dm_p(t)}{dt} \right]$$

$$m_p(t) = \frac{D_f}{D_p} \int_{-\infty}^{t} m_f(\sigma) d\sigma$$

Figure 5–8 Angle modulator circuits. RFC = radio-frequency choke.
Figure 5–8 Angle modulator circuits. RFC = radio-frequency choke.

Instantaneous Frequency

\[ s(t) = R(t) \cos \psi(t) \]
\[ = A_c \cos \psi(t) \quad \text{FM or PM} \]
\[ = A_c \cos(\omega_c t + \theta(t)) \]

The instantaneous frequency in Hz is:

\[ f_i(t) = \frac{1}{2\pi} \omega_i(t) = \frac{1}{2\pi} \left[ \frac{d\psi(t)}{dt} \right] = f_c + \frac{\dot{\theta}(t)}{2\pi} \]
FM and PM differences

PM: instantaneous phase deviation of the carrier phase is proportional to the amplitude of m(t)

\[ \theta(t) = D_p m(t) \text{ radians} \]

\[ D_p \text{ in } \frac{\text{radians}}{\text{volt}} \]

- Modulation Constant
- Modulation sensitivity
- Phase sensitivity

Instantaneous phase in radians: \( \psi(t) = \omega t + \theta(t) = \omega t + D_p m(t) \)

Instantaneous frequency in Hz: \( f(t) = \frac{d\psi(t)}{2\pi dt} = f_c + \frac{\dot{\theta}(t)}{2\pi} = f_c + \frac{D_p m(t)}{2\pi} \)

FM and PM differences

FM: instantaneous frequency deviation from the carrier frequency is proportional to m(t)

\[ \theta(t) = D_f \int_{-\infty}^{t} m(\alpha) d\alpha \text{ radians} \]

\[ D_f \text{ in } \frac{\text{radians}}{\text{volt - sec}} \]

The instantaneous phase in radians: \( \psi(t) = \omega t + \theta(t) = \omega t + D_f \int_{-\infty}^{t} m(\alpha) d\alpha \)

The instantaneous frequency in Hz: \( f(t) = \frac{d\psi(t)}{2\pi dt} = f_c + \frac{\dot{\theta}(t)}{2\pi} = f_c + \frac{D_f m(t)}{2\pi} \)
FM and PM differences

**FM**: instantaneous **frequency deviation** from the carrier frequency is proportional to \( m(t) \)

\[
f_d(t) \equiv f_i(t) - f_c = \frac{1}{2\pi} \theta(t) = \frac{1}{2\pi} D_f m(t)
\]

**Modulation Constants**

\[
\begin{align*}
D_p &= K_p \Rightarrow \frac{\text{radians}}{\text{volt}} \\
D_f &= K_f \Rightarrow \frac{\text{rad}}{\text{volt - sec}} = 2\pi \frac{\text{Hz}}{\text{volt}}
\end{align*}
\]

**FM**

frequency deviation \( \equiv f_d(t) = f_i(t) - f_c = \frac{1}{2\pi} \left[ \frac{d\theta(t)}{dt} \right] \)

peak frequency deviation \( \equiv \Delta F = \text{max} \left\{ \frac{1}{2\pi} \left[ \frac{d\theta(t)}{dt} \right] \right\} = \frac{1}{2\pi} D_f V_p \)

\[
V_p = \text{max}[m(t)]
\]

frequency modulation index \( \equiv \beta_f = \frac{\Delta F}{B} \) \( B \) is the bandwidth of \( m(t) \)
PM and digital modulation

phase deviation = $\theta(t)$
peak phase deviation = $\Delta \theta = \max[\theta(t)] = D_p V_p$

$V_p = \max[m(t)]$
phase modulation index = $\beta_p = \Delta \theta$

note:
when $m(t)$ is a sinusoidal signal set such that the PM and FM signals have the same peak frequency deviation, then $\beta_p = \beta_f$

For Digital signals the modulation index:

$h = \frac{2\Delta \theta}{\pi}$

where $2\Delta \theta$ is the pk - pk phase change in one symbol duration, $T_s$
Figure 5–9 FM with a sinusoidal baseband modulating signal.

(b) Instantaneous Frequency of the Corresponding FM Signal

FM from PM and PM from FM

(a) Generation of FM Using a Phase Modulator

(b) Generation of PM Using a Frequency Modulator
FM and PM with \( m(t) = \cos(2\pi f_m t) \)

Let \( m(t) = a \cos(2\pi f_m t) \)

For PM \( \phi(t) = k_p m(t) = k_p a \cos(2\pi f_m t) \),

For FM
\[
\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau = \frac{k_f a}{f_m} \sin(2\pi f_m t).
\]

\[
u(t) = \begin{cases} 
A_c \cos \left( 2\pi f_c t + k_p a \cos(2\pi f_m t) \right), & \text{PM} \\
A_c \cos \left( 2\pi f_c t + \frac{k_f a}{f_m} \sin(2\pi f_m t) \right), & \text{FM}
\end{cases}
\]

Define the modulation indices:
\[
\beta_p = k_p a, \quad \beta_f = \frac{k_f a}{f_m}.
\]
Define the modulation indices:

\[ \beta_p = k_p a \]
\[ \beta_p = k_p \max[|m(t)|] \]
\[ \beta_p = \Delta \phi_{\text{max}} \]
\[ \beta_f = \frac{k_f a}{f_m} , \]
\[ \beta_f = \frac{k_f \max[|m(t)|]}{W} \]
\[ \beta_f = \frac{\Delta f_{\text{max}}}{W} . \]
Spectrum Characteristics of FM

• FM/PM is exponential modulation

Let \[ \phi(t) = \beta \sin(2\pi f_m t) \]

\[ u(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \]

\[ = \text{Re} \left( A_c e^{j(2\pi f_c t + \beta \sin(2\pi f_m t))} \right) \]

\[ u(t) \text{ is periodic in } f_m \]
we may therefore use the Fourier series
Spectrum with Sinusoidal Modulation

\[ g(t) = e^{j\beta \sin(2\pi f_m t)} \]

u(t) is periodic in \( f_m \)
we may therefore use the Fourier series

\[ c_n = f_m \int_{0}^{\frac{1}{f_m}} e^{j\beta \sin 2\pi f_m t} e^{-jn2\pi f_m t} \, dt \]

\[ u = 2\pi f_m t \]
\[ \frac{1}{2\pi} \int_{0}^{2\pi} e^{j\beta (\sin u - nu)} \, du. \]

J_n Bessel Function

\[ e^{j\beta \sin 2\pi f_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}. \]

\[ u(t) = \text{Re} \left( A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t} e^{j2\pi f_c t} \right) \]

\[ = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos (2\pi (f_c + nf_m) t). \]
TABLE 5–2  
FOUR-PLACE VALUES OF THE BESSEL FUNCTIONS \( J_n(\beta) \)
TABLE 5–3  ZEROS OF BESSEL FUNCTIONS:
VALUES FOR $\beta$ WHEN $J_n(\beta) = 0$

<table>
<thead>
<tr>
<th>Order of Bessel Function, $n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ for 1st zero</td>
<td>2.40</td>
<td>3.83</td>
<td>5.14</td>
<td>6.38</td>
<td>7.50</td>
<td>8.77</td>
<td>9.93</td>
</tr>
<tr>
<td>$\beta$ for 2nd zero</td>
<td>5.52</td>
<td>7.02</td>
<td>8.42</td>
<td>9.76</td>
<td>11.06</td>
<td>12.34</td>
<td>13.59</td>
</tr>
<tr>
<td>$\beta$ for 3rd zero</td>
<td>8.65</td>
<td>10.17</td>
<td>11.62</td>
<td>13.02</td>
<td>14.37</td>
<td>15.70</td>
<td>17.00</td>
</tr>
<tr>
<td>$\beta$ for 4th zero</td>
<td>11.79</td>
<td>13.32</td>
<td>14.80</td>
<td>16.22</td>
<td>17.62</td>
<td>18.98</td>
<td>20.32</td>
</tr>
<tr>
<td>$\beta$ for 5th zero</td>
<td>14.93</td>
<td>16.47</td>
<td>17.96</td>
<td>19.41</td>
<td>20.83</td>
<td>22.21</td>
<td>23.59</td>
</tr>
<tr>
<td>$\beta$ for 6th zero</td>
<td>18.07</td>
<td>19.61</td>
<td>21.12</td>
<td>22.58</td>
<td>24.02</td>
<td>25.43</td>
<td>26.82</td>
</tr>
<tr>
<td>$\beta$ for 7th zero</td>
<td>21.21</td>
<td>22.76</td>
<td>24.27</td>
<td>25.75</td>
<td>27.20</td>
<td>28.63</td>
<td>30.03</td>
</tr>
<tr>
<td>$\beta$ for 8th zero</td>
<td>24.35</td>
<td>25.90</td>
<td>27.42</td>
<td>28.91</td>
<td>30.37</td>
<td>31.81</td>
<td>33.23</td>
</tr>
</tbody>
</table>

Figure 5–11 Magnitude spectra for FM or PM with sinusoidal modulation for various modulation indexes.
Figure 5–11 Magnitude spectra for FM or PM with sinusoidal modulation for various modulation indexes.

NBFM- Narrowband Frequency Modulation
WBFM - Wideband Frequency
Modulation Carson’s Bandwidth Rule

EELE445-14
Lecture 31
Narrowband FM

• Only the \( J_0 \) and \( J_1 \) terms are significant
• Same Bandwidth as AM
• Using Eulers identity, and \( \phi(t) \ll 1 \):

\[
u(t) = A_c \cos 2\pi f_c t \cos \phi(t) - A_c \sin 2\pi f_c t \sin \phi(t)
\approx A_c \cos 2\pi f_c t - A_c \phi(t) \sin 2\pi f_c t,
\]

Notice the sidebands are “\( \sin \)”, not “\( \cos \)” as in AM

Narrowband FM as a Phaser

(a)

(b)
Frequency Multiplication:
Wideband FM from Narrowband FM

\[ s_o(t) = \text{Re}(e^{j\phi(t)} e^{j2\pi f_m t})^n = \text{Re}(e^{jn\phi(t)} e^{j2\pi n f_m t}) \]

\[ n \phi(t) = nD_f \int_{-\infty}^{t} m(\lambda) d\lambda \]

\[ \beta_{f_{\text{out}}} = n \beta_f \text{min} \]

• The Output Carrier frequency = n \times f_c
• The output modulation index = n \times \beta_{fm}
• The output bandwidth increases according to Carson’s Rule

Effective Bandwidth- Carson’s Rule for Sine Wave Modulation

\[ B_c = 2(\beta + 1) f_m, \quad \text{where } \beta \text{ is the modulation index } f_m \text{ is the sinusoidal modulation frequency} \]

\[ m(t) = a \cos(2\pi f_m t). \]

\[ B_c = 2(\beta + 1) f_m = \begin{cases} 2(k_p a + 1) f_m, & \text{PM} \\ 2 \left( \frac{k_f a}{f_m} + 1 \right) f_m, & \text{FM} \end{cases} \]

• Notice for FM, if k_p a >> f_m, increasing fm does not increase B_c much
• B_c is linear with f_m for PM
Figure 5–11 Magnitude spectra for FM or PM with sinusoidal modulation for various modulation indexes.

(a) $\beta = 0.2$

(b) $\beta = 1.0$

(c) $\beta = 2.0$

(d) $\beta = 5.0$
When \( m(t) \) is a sum of sine waves

Consider now the case where \( m(t) \) is the sum of \( K \) separate sine waves. That is, let

\[
m(t) = \sum_{i=1}^{K} c_i \cos (\omega_i t + \theta_i)
\]

where \( c_i, \omega_i, \) and \( \theta_i \) are the corresponding deviations, frequency, and phase angles.
When \( m(t) \) is a sum of sine waves

\[
c(t) = A \cos \left[ \omega_c t + \sum_{i=1}^{\infty} \beta_i \sin (\omega_i t + \theta_i) + \psi \right]
\]

\[
c(t) = A \sum_{k=1}^{\infty} \cdots \sum_{k=1}^{\infty} \left[ \prod_{j=1}^{K} J_n(b_j) \right] \cos \left[ \omega_c t + \sum_{i=1}^{K} k_i(\omega_i t + \theta_i) + \psi \right]
\]

The preceding equation represents the general expression for the FM carrier modulated by \( K \) sinusoids. Note that it corresponds to a collection of harmonic frequencies at all the sidebands \( \sum_{i=1}^{K} k_i \omega_i \), where all combinations of integers for the \( \{k_i\} \) must be considered. Each such combination \( \{k_1, k_2, \ldots, k_K\} \) yields a different sinusoid, each with its own phase, \( \psi \), and its own amplitude \( \{ A m(\beta_k) \} \). In particular, we note that the carrier component of \( \omega_c \) corresponds to \( k_1 = k_2 = \cdots = k_K = 0 \) and has amplitude \( \{ A m(6) \} \), whereas the frequency component at frequency \( (\omega_c + \omega_m) \) corresponds to \( k_1 = 0, k_2 = 0, \ldots, k_1 = 1, k_{j+1} = 0, \ldots, k_K = 0 \) and has amplitude \( \{ A m(\beta_j) \} \). We also note that the component at \( (\omega_c + \omega_j) \) contains the exact phase angle of the \( j \)th sine wave in (2.4.14) added to that of the carrier.

### Sideband Power

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal Amplitude:</td>
<td>( A_c = 1V )</td>
</tr>
<tr>
<td>Modulating frequency:</td>
<td>( f_m = 1KHz )</td>
</tr>
<tr>
<td>Carrier peak deviation:</td>
<td>( \Delta f = 2.4KHz )</td>
</tr>
<tr>
<td>Modulation index:</td>
<td>( \beta = \frac{\Delta f}{f_m} )</td>
</tr>
<tr>
<td>Reference equation:</td>
<td>( x(t) = \sum_{n=-\infty}^{\infty} \left[ A_c \sin(n, \beta) \cos[n(\omega_c + \omega_m)] \right] )</td>
</tr>
<tr>
<td>Power in the signal:</td>
<td>( P_c = \frac{A_c^2}{2 \cdot 1\Omega} )</td>
</tr>
<tr>
<td>Carson's rule bandwidth:</td>
<td>( BW = 2(\beta + 1)f_m )</td>
</tr>
<tr>
<td>Order of significant sidebands predicted by Carson's rule:</td>
<td>( n = \text{round}(\beta + 1) )</td>
</tr>
<tr>
<td>Power as a function of number of sidebands:</td>
<td>( P_{\text{sum}}(k) = \sum_{n=-k}^{k} \left( \frac{A_c \sin[n(\beta)]}{2 \cdot 1\Omega} \right)^2 )</td>
</tr>
<tr>
<td>Percent of power predicted by Carson's rule:</td>
<td>( \frac{P_{\text{sum}}(n)}{P_c} = 99.118 )</td>
</tr>
</tbody>
</table>
**Power vs Bandwidth**

\[ \text{PERCENT OF TOTAL POWER} \]

\[ \frac{P_{\text{sum}(k)}}{P_c} \times 100 \]

\[ \begin{array}{c}
0 & 0.5 & 1 & 1.5 & 2 & 2.5 & 3 \\
0 & 50 & 100 & & & & \\
\end{array} \]

\[ k \]

\[ \text{Sideband Power } \beta = 2.4 \]

\[ k := 0 \ldots 10 \]

\[ J_k := J_n(k, \beta) \]

\[ p_k = (J_k)^2 \]

\[ \beta = 2.4 \]

\[ n = 3 \]

\[ \begin{array}{c|c}
0 & 0 \\
0 & 6.288 \times 10^{-6} \\
1 & 0.271 \\
2 & 0.186 \\
3 & 0.039 \\
4 & 4.135 \times 10^{-3} \\
5 & 2.638 \times 10^{-4} \\
6 & 1.134 \times 10^{-5} \\
7 & 3.513 \times 10^{-7} \\
8 & 8.237 \times 10^{-9} \\
9 & 1.513 \times 10^{-10} \\
10 & 2.238 \times 10^{-12} \\
\end{array} \]

\[ P_0 + 2 \sum_{j=1}^{n} P_j = 0.991 \]
Sideband Power $\beta=0.1$

\[
j := 0 \ldots 5 \quad \beta := 0.1 \quad n := 1
\]

\[
V_j := J_n(j, \beta) \quad U_j := (V_j)^2
\]

\[
V = \begin{pmatrix}
0.998 \\
0.05 \\
1.249 \times 10^{-3} \\
2.082 \times 10^{-5} \\
2.603 \times 10^{-7} \\
2.603 \times 10^{-9}
\end{pmatrix} \quad U = \begin{pmatrix}
0.995 \\
2.494 \times 10^{-3} \\
1.56 \times 10^{-6} \\
4.335 \times 10^{-10} \\
6.775 \times 10^{-14} \\
0
\end{pmatrix}
\]

\[
U_0 + 2 \sum_{j=1}^{n} U_j = 1
\]

\[
X := \begin{pmatrix}
0.832 \\
0.082 \\
1.907 \times 10^{-3} \\
1.936 \times 10^{-5} \\
1.099 \times 10^{-7} \\
3.979 \times 10^{-10}
\end{pmatrix}
\]

\[
X_0 + 2 \sum_{j=1}^{n} X_j = 0.996
\]

Sideband Power $\beta=0.6$

\[
\beta := 0.6 \quad n := 1
\]

\[
W_j := J_n(j, \beta) \quad X_j := (W_j)^2
\]

\[
W = \begin{pmatrix}
0.912 \\
0.287 \\
0.044 \\
4.4 \times 10^{-3} \\
3.315 \times 10^{-4} \\
1.995 \times 10^{-5}
\end{pmatrix} \quad X = \begin{pmatrix}
0.832 \\
0.082 \\
1.907 \times 10^{-3} \\
1.936 \times 10^{-5} \\
1.099 \times 10^{-7} \\
3.979 \times 10^{-10}
\end{pmatrix}
\]

\[
X_0 + 2 \sum_{j=1}^{n} X_j = 0.996
\]
Modulation index $M = \frac{x}{10}$

$n := \text{round}(M + 1)$

$2 \times n$ is the number of significant sidebands per Carson's rule

$n = 9$

Bandwidth: $2 \times F_m$

Modulation index: $M$

FM/PM modulation index set to $\pi/2$ for peak phase deviation of $\pi/2$

set to $\Delta f/F_m$ for frequency modulation. Spectrum is the same for sine wave modulation.

$\beta = 0.4$, Sideband Level $= \beta/2$ for Narrowband FM
\[ \beta = 0.9, \text{ Sideband Level } = \beta/2 \text{ for Narrowband FM} \]

\[ \beta = 2.4, \text{ Carrier Null} \]
\( \beta = 3.8, \) first sideband null

\[ \text{Bessel Functions} \]

\[ \text{Single Sided Spectrum} \]

\( \beta = 5.1, \) second sideband null

\[ \text{Bessel Functions} \]

\[ \text{Single Sided Spectrum} \]
Power vs BW, $\beta=0.1$

The second term includes power in $+Jn$ and power in $-Jn$, i.e., the upper and lower sideband pairs.

Power vs BW, $\beta=0.9$

- M = 0.9
- Fm = 1 Hz
- Bandwidth = 4 Hz
- $\frac{P(M, k)}{\Delta^2} = 100 - 99.958$
Power vs BW, $\beta=2.4$

$P(M,k) = \frac{A^2}{2} \cdot 100 / 99.945$

$M = 2.4$

$F_m = 1$ Hz

Bandwidth = 8 Hz

Montana State University

Mountains of Minds