# EELE445-14

LECTURE 6

Power and Energy in the Frequency Domain



## Energy and Power in a Waveform

**Parseval's Theorem and Energy Spectral Density**

**Parseval's Theorem.**

$$\int_{-\infty}^{\infty} w_1(t)w_2^*(t)\,dt = \int_{-\infty}^{\infty} W_1(f)W_2^*(f)\,df \quad (2\text{–}40)$$

If $w_1(t) = w_2(t) = w(t)$, then the theorem reduces to

**Rayleigh's energy theorem,** which is

$$E = \int_{-\infty}^{\infty} |w(t)|^2\,dt = \int_{-\infty}^{\infty} |W(f)|^2\,df \quad (2\text{–}41)$$

DEFINITION. The *energy spectral density (ESD)* is defined for energy waveforms by

$$\mathcal{E}(f) = |W(f)|^2 \quad (2\text{–}42)$$

where $w(t) \leftrightarrow W(f)$. $\mathcal{E}(f)$ has units of joules per hertz.

Using Eq. (2–41), we see that the total normalized energy is given by the area under the ESD function:

$$E = \int_{-\infty}^{\infty} \mathcal{E}(f)\,df \quad (2\text{–}43)$$



## Energy and Power in a Waveform

- Energy Spectral Density, **ESD**

$$E(f) == |W(f)|^2$$

- Power Spectral Density, **PSD** for a truncated waveform of duration T

$$w_T(t) = \begin{cases} w(t), & -T/2 < t < T/2 \\ 0, & t \text{ elsewhere} \end{cases} = w(t)\Pi\left(\frac{t}{T}\right) \quad (2\text{–}64)$$

Using Eq. (2–13), we obtain the average normalized power:

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w^2(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} w_T^2(t)\,dt$$

By the use of Parseval's theorem, Eq. (2–41), the average normalized power becomes

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |W_T(f)|^2\,df = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{|W_T(f)|^2}{T}\right) df \quad (2\text{–}65)$$



## Energy and Power in a Waveform

DEFINITION. The *power spectral density (PSD)* for a deterministic power waveform is†

$$\mathcal{P}_w(f) \triangleq \lim_{T\to\infty} \left(\frac{|W_T(f)|^2}{T}\right) \quad (2\text{–}66)$$

where $w_T(t) \leftrightarrow W_T(f)$ and $\mathcal{P}_w(f)$ has units of watts per hertz.

Note that the PSD is always a real nonnegative function of frequency. In addition, the PSD is not sensitive to the phase spectrum of $w(t)$, because that is lost due to the absolute value operation used in Eq. (2–66). From Eq. (2–65), the normalized average power is‡

$$P = \langle w^2(t)\rangle = \int_{-\infty}^{\infty} \mathcal{P}_w(f)\,df \quad (2\text{–}67)$$

That is, the area under the PSD function is the normalized average power.

## Time Average Autocorrelation Function

**Definition.** The *autocorrelation* of a real (physical) waveform is§

$$R_w(\tau) \triangleq \langle w(t)w(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w(t)w(t+\tau)\, dt \qquad (2\text{–}68)$$

Furthermore, it can be shown that the PSD and the autocorrelation function are Fourier transform pairs; that is,

$$R_w(\tau) \leftrightarrow \mathcal{P}_w(f) \qquad (2\text{–}69)$$

where $\mathcal{P}_w(f) = \mathcal{F}[R_w(\tau)]$. This is called the *Wiener–Khintchine theorem.* The theorem, along with properties for $R(\tau)$ and $\mathcal{P}(f)$, are developed in Chapter 6.

In summary, the PSD can be evaluated by either of the following two methods:

1. *Direct method*, by using the definition, Eq. (2-66).◊
2. *Indirect method*, by first evaluating the autocorrelation function and then taking the Fourier transform: $\mathcal{P}_w(f) = \mathcal{F}[R_w(\tau)]$.

$$P = \langle w^2(t)\rangle = W_{\text{rms}}^2 = \int_{-\infty}^{\infty} \mathcal{P}_w(f)\, df = R_w(0) \qquad (2\text{–}70)$$



## Autocorrelation of a Sinusoid

**Example 2–10 PSD of a Sinusoid**

Let

$$w(t) = A \sin \omega_0 t$$

The PSD will be evaluated using the indirect method. The autocorrelation is

$$R_w(\tau) = \langle w(t)w(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A^2 \sin \omega_0 t \sin \omega_0(t+\tau)\, dt$$

Using a trigonometric identity, from Appendix A we obtain

$$R_w(\tau) = \frac{A^2}{2}\cos \omega_0\tau \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt - \frac{A^2}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos(2\omega_0 t + \omega_0\tau)\, dt$$

which reduces to

$$R_w(\tau) = \frac{A^2}{2}\cos \omega_0\tau \qquad (2\text{–}71)$$



## Autocorrelation of a Sinusoid

The PSD is then

$$\mathcal{P}_w(f) = \mathcal{F}\left[\frac{A^2}{2}\cos \omega_0\tau\right] = \frac{A^2}{4}[\delta(f-f_0) + \delta(f+f_0)] \qquad (2\text{–}72)$$

as shown in Fig. 2–9. The PSD may be compared to the "voltage" spectrum for a sinusoid found in Example 2–4 and shown in Fig. 2–4.

The average normalized power may be obtained by using Eq. (2–67):

$$P = \int_{-\infty}^{\infty} \frac{A^2}{4}[\delta(f-f_0) + \delta(f+f_0)]\, df = \frac{A^2}{2} \qquad (2\text{–}73)$$

This value, $A^2/2$, checks with the known result for the normalized power of a sinusoid:

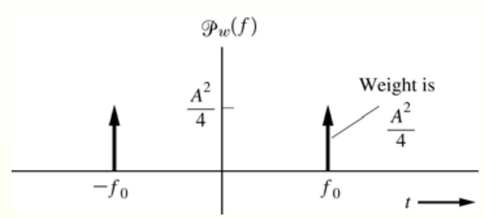
$$P = \langle w^2(t)\rangle = W_{\text{rms}}^2 = (A/\sqrt{2})^2 = A^2/2 \qquad (2\text{–}74)$$

It is also realized that $A \sin \omega_0 t$ and $A \cos \omega_0 t$ have exactly the same PSD (and autocorrelation function) because the phase has no effect on the PSD. This can be verified by evaluating the PSD for $A \cos \omega_0 t$, using the same procedure that was used earlier to evaluate the PSD for $A \sin \omega_0 t$.



## Autocorrelation of a Sinusoid

**Figure 2–9** Power spectrum of a sinusoid.



# Time Average Autocorrelation Function

**DEFINITION.** The *autocorrelation* of a real (physical) waveform is[§]

$$R_w(\tau) \triangleq \langle w(t)w(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w(t)w(t+\tau)\,dt \quad (2\text{–}68)$$

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$$P = \langle w^2(t)\rangle = W_{\text{rms}}^2 = \int_{-\infty}^{\infty} \mathcal{P}_w(f)\,df = R_w(0) \quad (2\text{–}70)$$





# Power and Power Spectra of a Waveform in a given Bandwidth

$P(f)$ is the double - sided power spectral density(textbook $c_n$)
$S(f)$ is the single - sided power spectral density, as seen on a spectrum analyzer.

$$S(f) = P(f) + P(-f) = 2P(f) \quad \frac{watts}{Hz}$$

The total power in a frequency range is :

$$P_{bw} = \int_{f_1}^{f_2} S(f)\,df \quad watts$$





**Figure 2–12** Periodic rectangular wave used in Example 2–12.

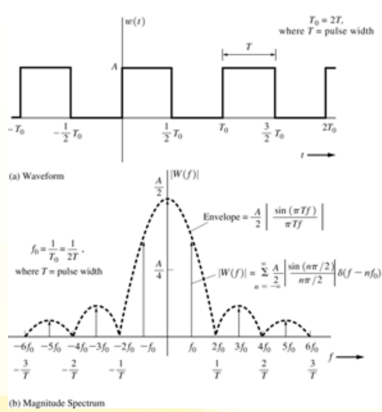
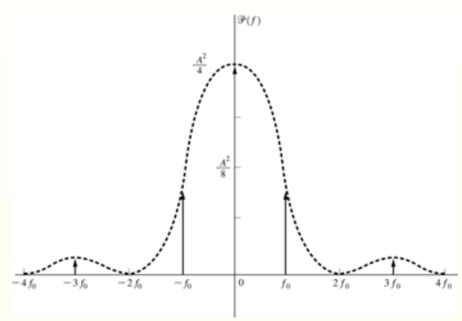









**Figure 2–13** PSD for a square wave used in Example 2–13.



## Orthogonal Functions

**Definition.** Functions $\varphi_n(t)$ and $\varphi_m(t)$ are said to be *orthogonal* with respect to each other over the interval $a < t < b$ if they satisfy the condition

$$\int_a^b \varphi_n(t)\varphi_m^*(t)\,dt = 0, \quad \text{where } n \neq m \tag{2-77}$$

Furthermore, if the functions in the set $\{\varphi_n(t)\}$ are orthogonal, then they also satisfy the relation

$$\int_a^b \varphi_n(t)\varphi_m^*(t)\,dt = \begin{Bmatrix} 0, & n \neq m \\ K_n, & n = m \end{Bmatrix} = K_n\delta_{nm} \tag{2-78}$$



## Normalized Orthogonal Series

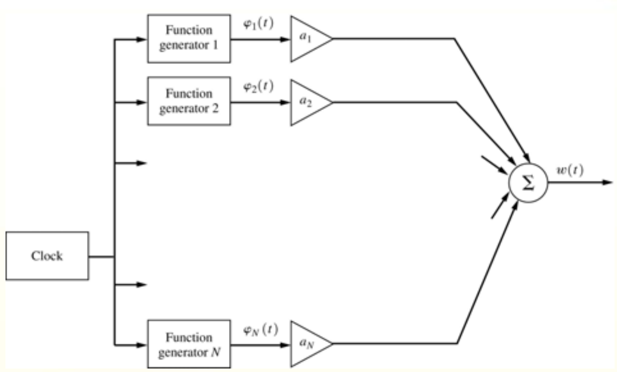
$$w(t) = \sum_n a_n\varphi_n(t)$$

$$where \;\; a_n = \frac{1}{K_n}\int_a^b w(t)\varphi_n^*(t)dt$$

$$K_n = \int_a^b \varphi_n(t)\varphi_n^*(t)dt$$ is the normalization constant



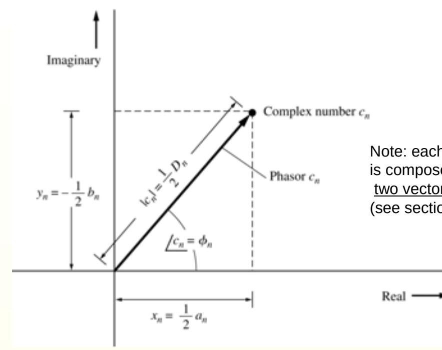
## Waveform synthesis using orthogonal functions.







## **Figure 2–11** Fourier series coefficients




Note: each physical frequency is composed of two vectors: $c_n$ and $c_{-n}$ (see section 2-5 in text)

### Fourier series coefficients

Polar Form:

$$w(t) = D_0 + \sum_{n=1}^{n=\infty} D_n \cos(n\omega_0 t + \varphi_n) \quad eq2-103$$

Complex form:

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad eq2-88$$

Quadrature form:

$$= \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \quad eq2-95$$




### Normalized Power of a Spectral component in Fourier Series

$$P_0 = \frac{D_0^2}{2} = c_0^2 = a_0^2 \Rightarrow \text{DC power}$$

$$P_n = \frac{D_n^2}{2} = 2|c_n|^2 = \frac{a_n^2}{2} + \frac{b_n^2}{2}$$

$$D_0 + \sum_{n=1}^{n=\infty} D_n \cos(n\omega_0 t + \varphi_n) \quad eq2-103 \text{ (single sided)}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad eq2-88 \text{ (double sided)}$$

$$= \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \quad eq2-95 \text{ (single sided)}$$

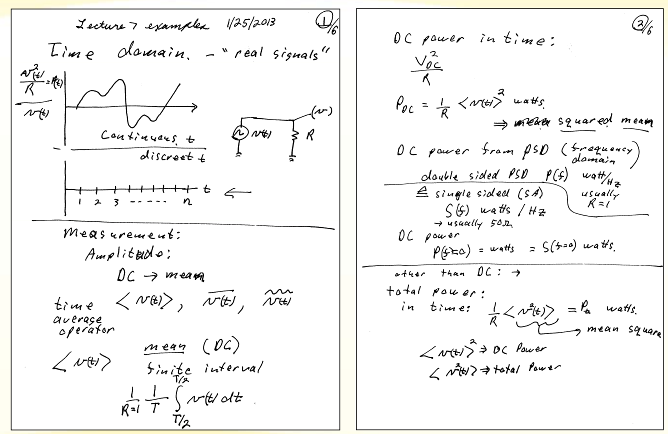


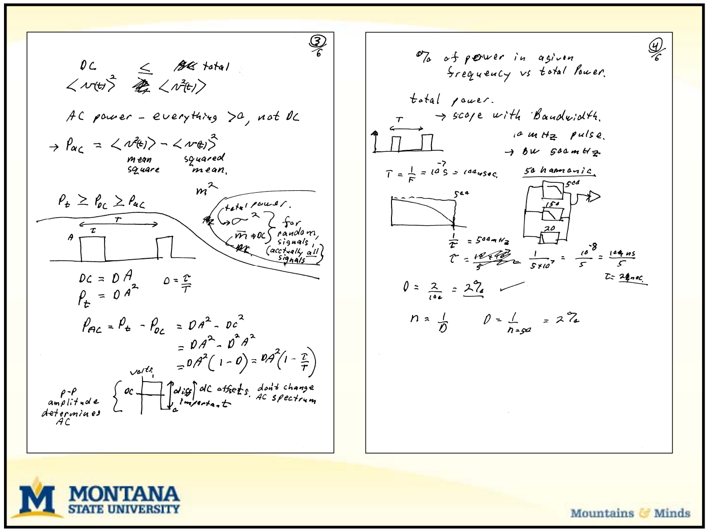

# EELE445-14

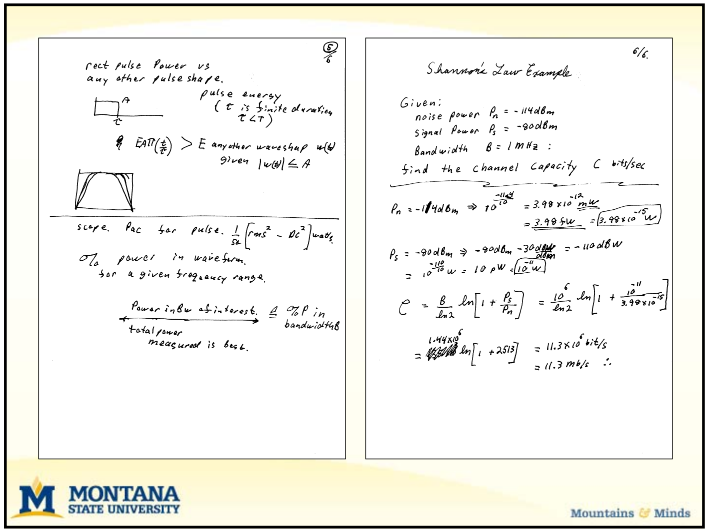
LECTURE 7

Some examples in class…






3/6
DC ≤ total
AC power - everything > 0, not DC
→ Pac = <v²(t)> - <v(t)>²
mean square
squared mean.
m²
Pt ≥ PDC ≥ PAC
total power.
σ²
m̄ ⇒ DC
for random, signals, (actually all signals)
DC = DA
Pt = DA²
D = τ/T
PAC = Pt - PDC = DA² - DC²
= DA² - D²A²
= DA²(1 - D) = DA²(1 - τ/T)
volts
p-p amplitude determines AC
DC
diff important
DC offsets. don't change AC spectrum
4/6
% of power in a given frequency vs total Power.
total power.
→ scope with Bandwidth.
10 MHz pulse.
→ BW 500 MHz
T = 1/F = 10⁻⁷ S = 100 nsec.
50 harmonic.
500
150
20
1/τ = 500 MHz
τ = 1/(5×10⁷) = 10⁻⁸/5 = 100 ns/5
τ = 20 nsec.
D = 2/100 = 2%
n = 1/D
D = 1/(n=50) = 2%


5/6
rect pulse Power vs any other pulse shape.
pulse energy
(τ is finite duration τ < T)
E AΠ(t/τ) > E any other waveshape w(t)
given |w(t)| ≤ A
scope. Pac for pulse. 1/50 [rms² - DC²] watts.
% power in waveform.
for a given frequency range.
Power in BW of interest.
total power measured is best.
≜ %P in bandwidth B
6/6.
Shannon's Law Example
Given:
noise power Pn = -114dBm
Signal Power Ps = -80dBm
Bandwidth B = 1 MHz :
find the channel capacity C bits/sec
Pn = -114dBm ⇒ 10^(-11.4/10) = 3.98×10⁻¹² mW
= 3.98 fW = 3.98×10⁻¹⁵ W
Ps = -80dBm ⇒ -80dBm - 30 dBW/dBm = -110dBW
= 10^(-110/10) W = 10 pW = 10⁻¹¹ W
C = B/ln2 ln[1 + Ps/Pn] = 10⁶/ln2 ln[1 + 10⁻¹¹/3.98×10⁻¹⁵]
= 1.44×10⁶ ln[1 + 2513] = 11.3×10⁶ bit/s
= 11.3 mb/s ∴