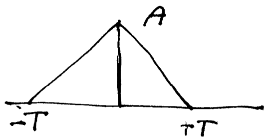
EELE445-15

Lecture 8

Example of Fourier Series for a Triangle from the Fourier Transform…

Homework password is:
15445

Fourier Series Example 1/28/2013 (1)



$A = 2$

$T = \frac{T_0}{2} = 1$

⇓ Table 2-2 p64

$\Lambda\left(\frac{t}{T}\right) \Rightarrow T\,Sa(\pi f T)^2 \qquad T_0 = 2$

$A\Lambda\left(\frac{t}{T_0/2}\right) = \frac{AT_0}{2}\,Sa\left(\frac{\pi f T_0}{2}\right)^2$

replace $f = \frac{n}{T_0}$, ~~...~~ scale by $\frac{1}{T_0}$

$\frac{1}{T_0}\left[\frac{AT_0}{2}\,Sa\left(\frac{\pi n T_0}{T_0\,2}\right)^2\right] = \frac{A}{2}\,Sa\left(\frac{\pi n}{2}\right)^2 \qquad A = 2.$

$C_n = \frac{2}{2}\,Sa\left(\frac{\pi n}{2}\right)^2 = \frac{\sin^2(\pi n/2)}{\left(\frac{\pi n}{2}\right)^2} = \frac{4}{(\pi n)^2}\sin^2\left(\frac{\pi n}{2}\right)$

use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\frac{4}{(\pi n)^2}\,\frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{2}\right)\right] = \frac{2}{(\pi n)^2}\left[1 - (-1)^n\right] \qquad n = 1, 2, 3 \ldots$

$= \frac{4}{(n\pi)^2}\begin{cases} n \text{ odd} \\ 0 = n \text{ even}\end{cases}$

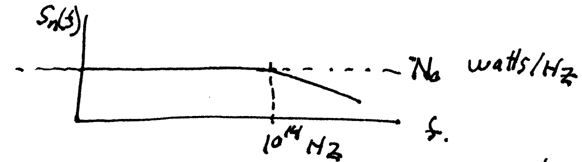
by inspection $C_0 = 0$

2

Continuous spectrum

example: white ~~dB~~ or Johnson (thermal noise).

single sided spectrum.



white ≜ constant power per Hz

~~density~~

watts/Hz ⇒ dBm/Hz.

volts/$\sqrt{Hz}$

$S_n(f) = N_0$ a constant.

$N_0 = S_n(f) = kT$ — K Boltzmans constant, T Kelvin.

at room temp

$S_n(f) = -174$ dBm/Hz

$= 10^{-20.4}$ watts/Hz $= 3.98 \times 10^{-21}$ watts/Hz

total noise power in 1 mHz BW?

$P_n = \int_{BW} S_n(f)\,df.$

3

Given $S_n(f) = N_o \quad 0 \le f \le 10^{14}\,Hz$:

$$P_n = \int_0^{10^6 = B} N_o\,df = N_o f \Big|_0^{B = 10^6} = \underline{\underline{N_o B}}$$

$$= 3.98 \times 10^{-21} \cdot 10^6$$

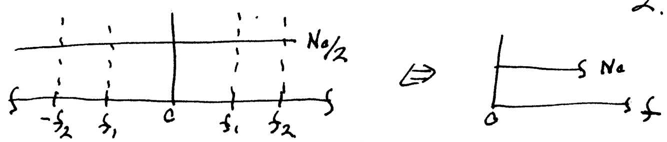
$$= 3.98 \times 10^{-15} \cong \underline{4\,FW} = 3.98 \times 10^{-12}\,mW = -114.002\,dBm$$

---

density: single sided → SA measures

vs

double sided ⇒ $\frac{N_o}{2}$



$$P_n = \int_{-f_2}^{-f_1} \frac{N_o}{2}\,df + \int_{f_1}^{f_2} \frac{N_o}{2}\,df = \frac{N_o}{2}(f_2 - f_1) + \frac{N_o}{2}(f_2 - f_1)$$

$$= N_o(f_2 - f_1) \triangleq N_o B$$

$$\underline{B \triangleq f_2 - f_1}$$

---

noise in dB land.

$N_o = KT$

$P_n = KTB$

$P_{n\,dBm} = 10\log(KT \cdot 1000 \cdot B) = 10\log(KT10^3) + 10\log B_{Hz}$

⇓

$P_{n\,dBm} = -174\,dBm/Hz + 10\log B$

$B = 10^6 \qquad -174 + 10\log 10^6 = \underline{-114\,dBm \quad B = 1\,MHz}$

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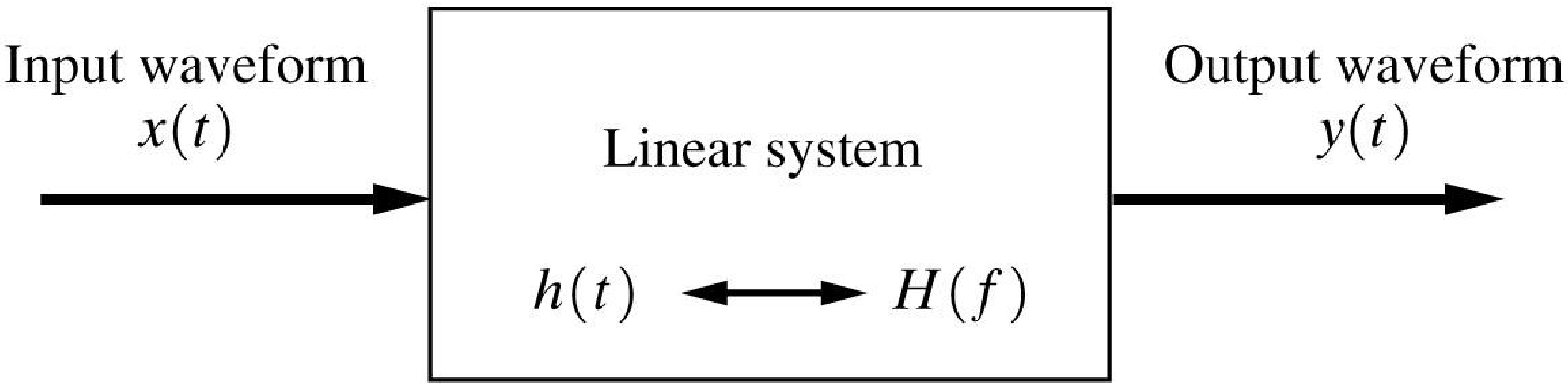
Lecture 8

LTI Systems

and

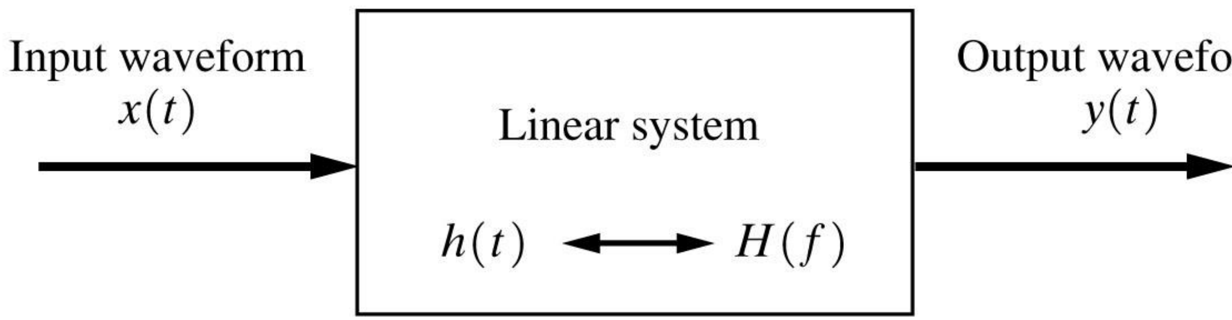
Filters

LTI System



Some descriptions for the input		Some descriptions for the output
$X(f)$	"Voltage" spectrum	$Y(f)$
$R_x(\tau)$	Autocorrelation function	$R_y(\tau)$
$\mathcal{P}_x(f)$	Power spectral density	$\mathcal{P}_y(f)$

Linear Time-Invariant System



Definition of a Linear System:

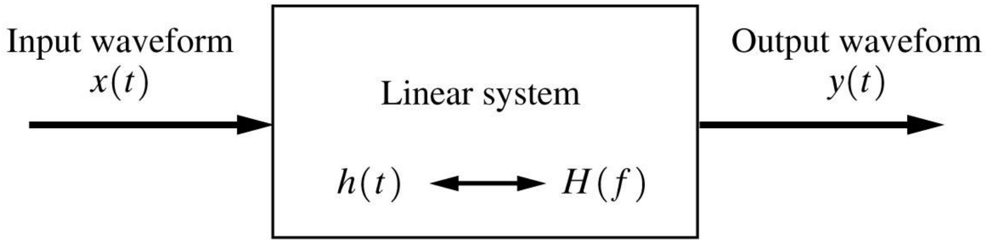
$$for\ L[\bullet], the\ linear\ differential\ equation\ system\ operator:$$

$$y(t) = L[ax(t) + b(z(t)] = aL[x(t)] + aL[z(t)]$$

Time Invariant : the delayed output $y(t - t_0)$ *is delayed by the same amount as the input,* $x(t - t_0)$*.*

The shape of the output waveform is independant of when the input waveform is applied to the system.

LTI System



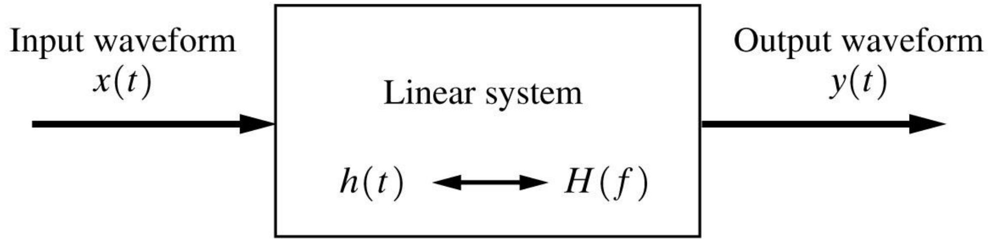
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)y^*(t-\tau)d\tau$$ (time avg or power autocor)

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\int_{-\infty}^{\infty} h(u)x(t-u)du\right]\left[\int_{-\infty}^{\infty} h(v)x^*(t-v)dv\right]dt$$

$P_y(f) \Rightarrow F\left[R_y(\tau)\right]$ PSD of the output spectrum

LTI System output Power and Voltage Spectrum



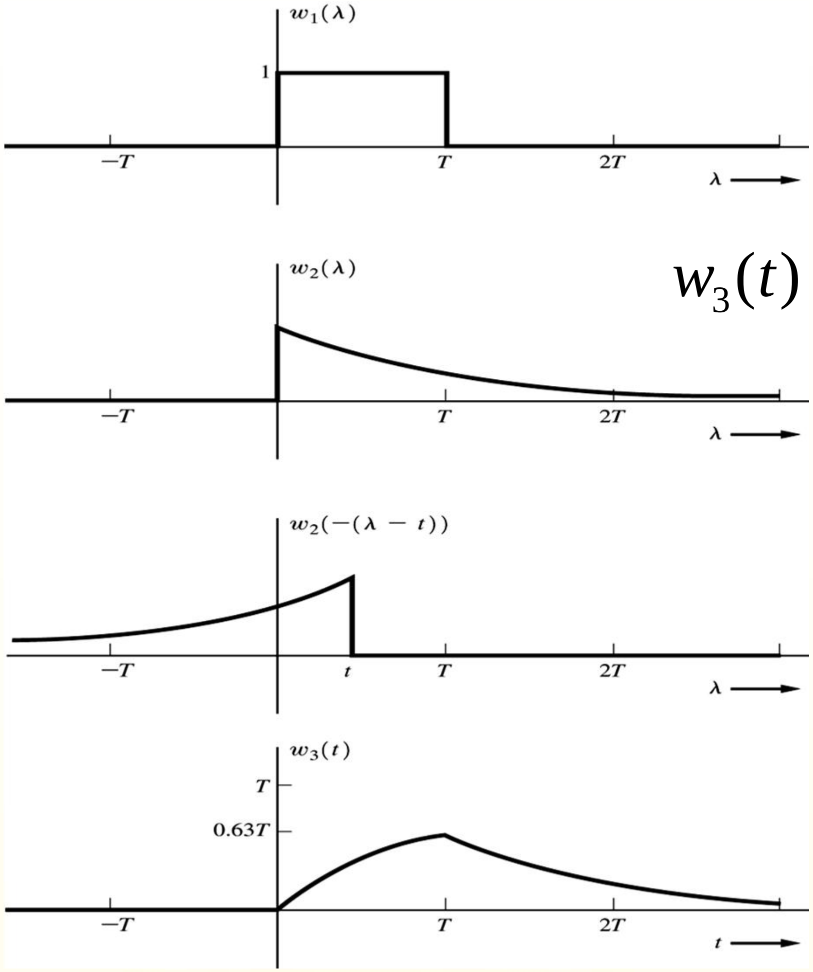
$$P_y(f) \Rightarrow F\lfloor R_y(\tau) \rfloor$$

$$P_y(f) = P_x(f)\left|H(f)\right|^2$$

$$y(t) = x(t) * h(t)$$

$$y(f) = X(f)H(f)$$

Figure 2–7 Convolution of a rectangle and an exponential.



$$w_3(t) = w_1(t) * w_2(t) \equiv \int_{-\infty}^{\infty} w_1(\lambda) w_2(t-\lambda) d\lambda$$

Power Autocorrelation: R(0)= power

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)x^*(t-\tau)\,dt.$$

$$\mathcal{S}_x(f) = \mathscr{F}[R_x(\tau)].$$

$$P_x = R_x(0) = \int_{-\infty}^{\infty} \mathcal{S}_x(f)\,df.$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 dt = R_x(0).$$

Power Signal Through a Filter:

By making a change of variables $w = t - u$ and changing the order of integration, we obtain

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h^*(v)$$

$$\times \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}-u}^{\frac{T}{2}+u} \left[x(w)x^*(u + w - \tau - v)\, dw\right] du\, dv$$

$$\stackrel{\text{a}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau + v - u)h(u)h^*(v)\, du\, dv$$

$$\stackrel{\text{b}}{=} \int_{-\infty}^{\infty} [R_x(\tau + v) \star h(\tau + v)]\, h^*(v)\, dv$$

$$\stackrel{\text{c}}{=} R_x(\tau) \star h(\tau) \star h^*(-\tau),$$

$$\mathcal{S}_y(f) = \mathcal{S}_x(f)H(f)H^*(f)$$

$$= \mathcal{S}_x(f)|H(f)|^2.$$

Periodic Signal Through a Filter: Time Average Autocorrelation

$$R_x(\tau) = Lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)dt = Lim_{k\to\infty} \frac{1}{kT_o} \int_{-\frac{kT_o}{2}}^{\frac{kT_o}{2}} x(t)x^*(t-\tau)dt$$

$$= Lim_{k\to\infty} \frac{k}{kT_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t-\tau)dt$$

$$R_x(\tau) = \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t-\tau)dt$$ for a periodic signal

Periodic Signal Through a Filter: Time Average Autocorrelation

$$R_x(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{+\infty} x_n x_m^* e^{j2\pi\frac{m}{T_0}\tau} e^{j2\pi\frac{n-m}{T_0}t}\,dt.$$

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} e^{j2\pi\frac{n-m}{T_0}t}\,dt = \delta_{mn},$$

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi\frac{n}{T_0}\tau}.$$

$$\mathcal{S}_x(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right).$$

$$P_x = \sum_{n=-\infty}^{\infty} |x_n|^2.$$

Periodic Signal Through a Filter Time Average Autocorrelation Through a LTI System

$$\mathcal{S}_y(f) = |H(f)|^2 \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right)$$

$$= \sum_{n=-\infty}^{\infty} |x_n|^2 \left|H\left(\frac{n}{T_0}\right)\right|^2 \delta\left(f - \frac{n}{T_0}\right)$$

$$P_y = \sum_{n=-\infty}^{\infty} |x_n|^2 \left|H\left(\frac{n}{T_0}\right)\right|^2.$$

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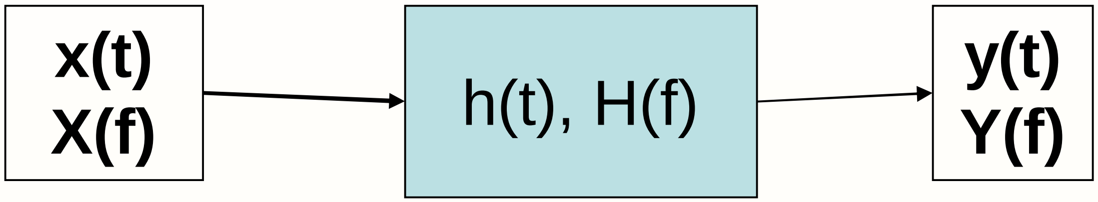
Lecture 9

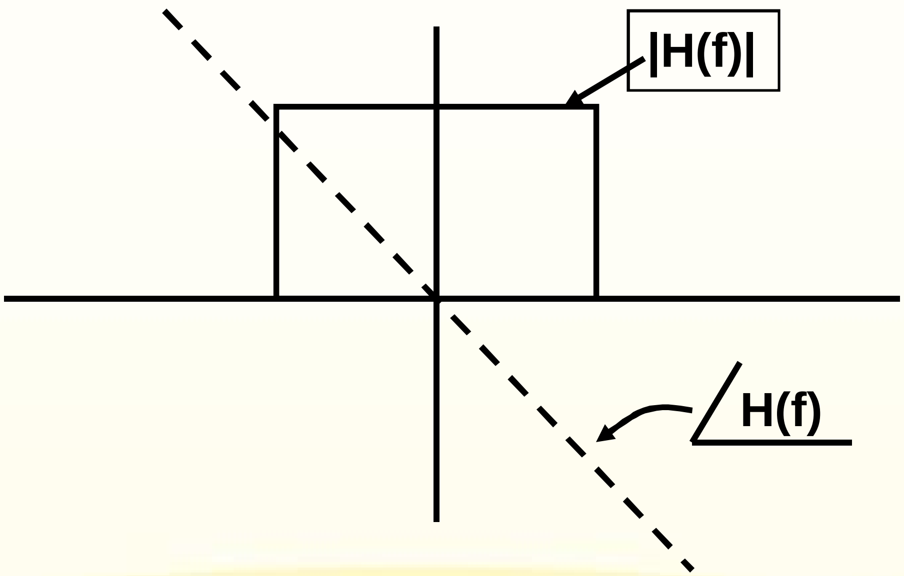
LTI Systems and Filters

Filters-Applications

- Control the Modulation Signal Bandwidth
  - Ex. Null bandwidth of rectangular pulse
  - Limits Transmission Bandwidth for Spectrum Conservation
- Optimize the SNR at the receiver input
- Waveform Shaping: $y(t)=x(t)*h(t)$
- Matched detection

Ideal Filter Response





Linear phase:

$$\theta(\omega) = \theta(0) - \tau_d \bullet \omega$$

Ideal Filter-Magnitude, Phase/Delay

For a transfer function H(s), at real frequencies, with s=jω,

$$H(j2\pi f) = |H(j2\pi f)| \bullet e^{j\theta(j2\pi f)} = G(\omega) \bullet e^{j\theta(\omega)}$$

Where G(ω) and θ(ω) are the gain and phase components.

Ideal Filter-Magnitude, Phase, Delay

$$\text{Phase or time delay} \equiv Pd(f) = \frac{-\theta(f)}{2\pi f} = \frac{-\theta(\omega)}{\omega}$$

$$\text{Group Delay} \equiv \tau_d(f) = -\frac{\partial\theta(f)}{\partial 2\pi f} = -\frac{\partial\theta(\omega)}{\partial\omega}$$

$$\text{Linear Phase} \equiv \theta(0) - \tau_d \bullet 2\pi f$$

Ideal Filter-Magnitude, Phase/Delay

- Both Pd(f) and $\tau_d(f)$ are functions of frequency

- Phase delay Pd(f) is the absolute delay and is of little significance

- Group Delay $\tau_d(f)$ is used as the criterion to evaluate phase nonlinearity. Group Delay is constant for all frequencies in an ideal filter.

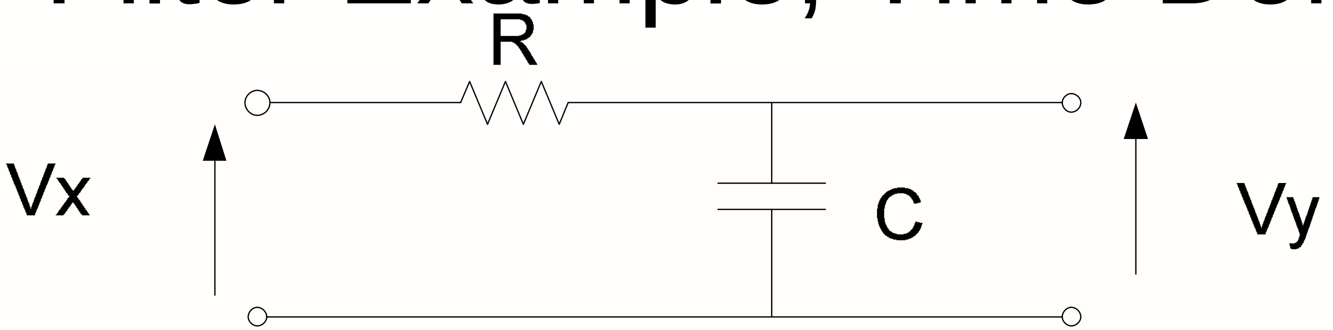
Ideal Filter-Magnitude, Phase/Delay

- Linear phase variation with frequency (over a band of frequencies) implies a constant Group Delay –no phase distortion in that band of frequencies
- In order to preserve the integrity of a pulse x(t), it is mandatory that the Group Delay of the system be constant up to the maximum frequency component of the pulse. This implies equal time delay for all frequencies of interest.

http://en.wikipedia.org/wiki/Butterworth_filter

Linear analog electronic filters
Butterworth filter
Chebyshev filter
Elliptic (Cauer) filter
Bessel filter
Gaussian filter
Optimum "L" (Legendre) filter

Filter Example, Time Domain:



$$H(s) = \frac{Vx}{Vy}$$

$$H(s) := \frac{1}{1 + R \cdot C \cdot s}$$

$$h(t) = \frac{1}{R \cdot C} \cdot e^{\frac{-t}{R \cdot C}} = \frac{1}{\tau} \cdot e^{\frac{-t}{\tau}}$$

Filter Example, Frequency Domain:

Let s=j2πf=jω

$$H(\omega) = \frac{1}{1 + \frac{j\cdot\omega}{\omega_o}} \qquad \omega_o := \frac{1}{R\cdot C}$$

$$H(f) = \frac{1}{1 + \frac{j\cdot f}{f_o}} \qquad f_o := \frac{1}{2\pi\cdot R\cdot C}$$

$$\theta(\omega) = \tan^{-1}(\omega\cdot R\cdot C) = \tan^{-1}\left(\frac{\omega}{\omega_o}\right)$$

$$\theta(f) = \tan^{-1}\left(\frac{f}{f_o}\right)$$

Filter Example:

The Group delay of the RC low pass is:

$$\tau_d(\omega) = -\frac{d\tau_d(\omega)}{d\omega} = \frac{\omega_0}{\omega^2 + \omega_0{}^2}$$

$$\tau_d(f) \quad = \frac{f_0}{f^2 + f_0{}^2}$$

Figure 2–15 Characteristics of an *RC* low-pass filter.

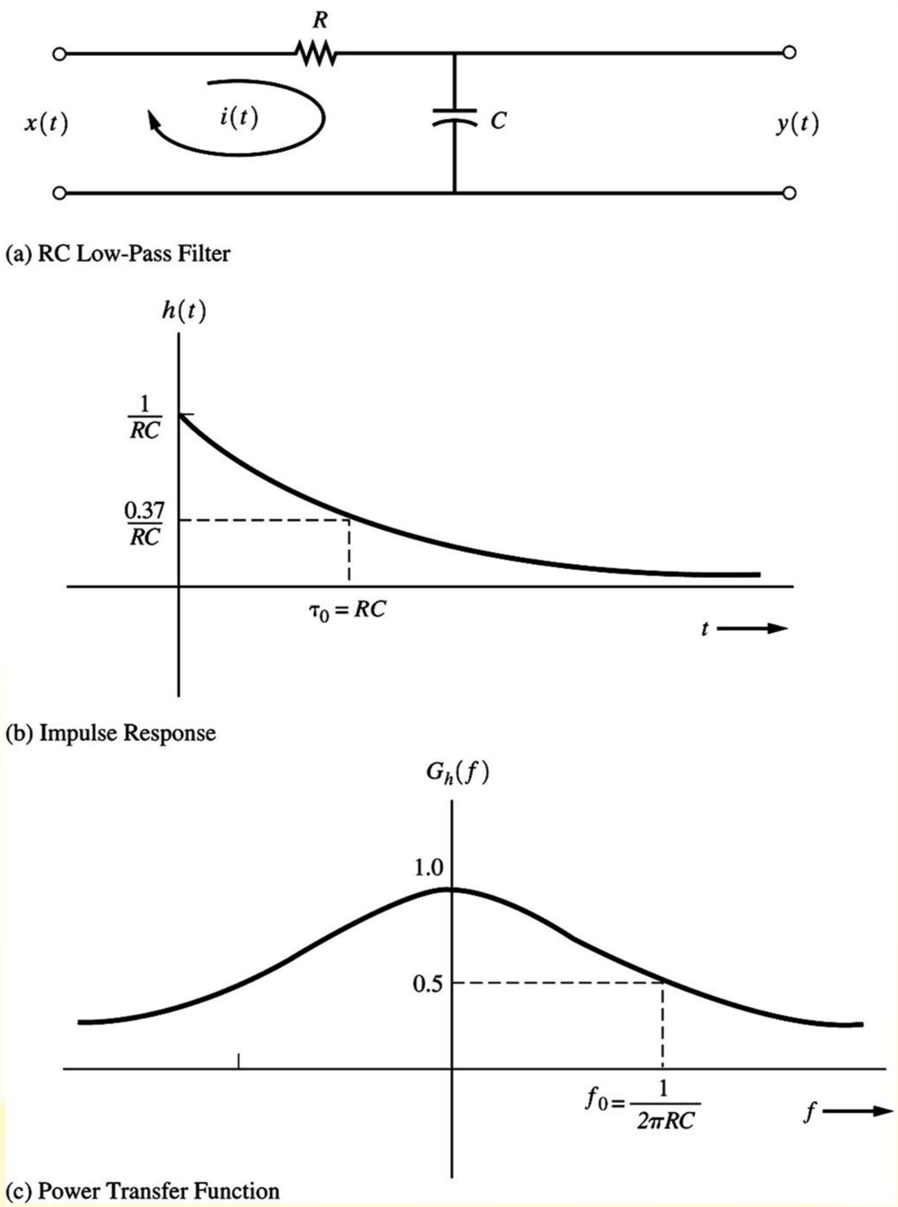
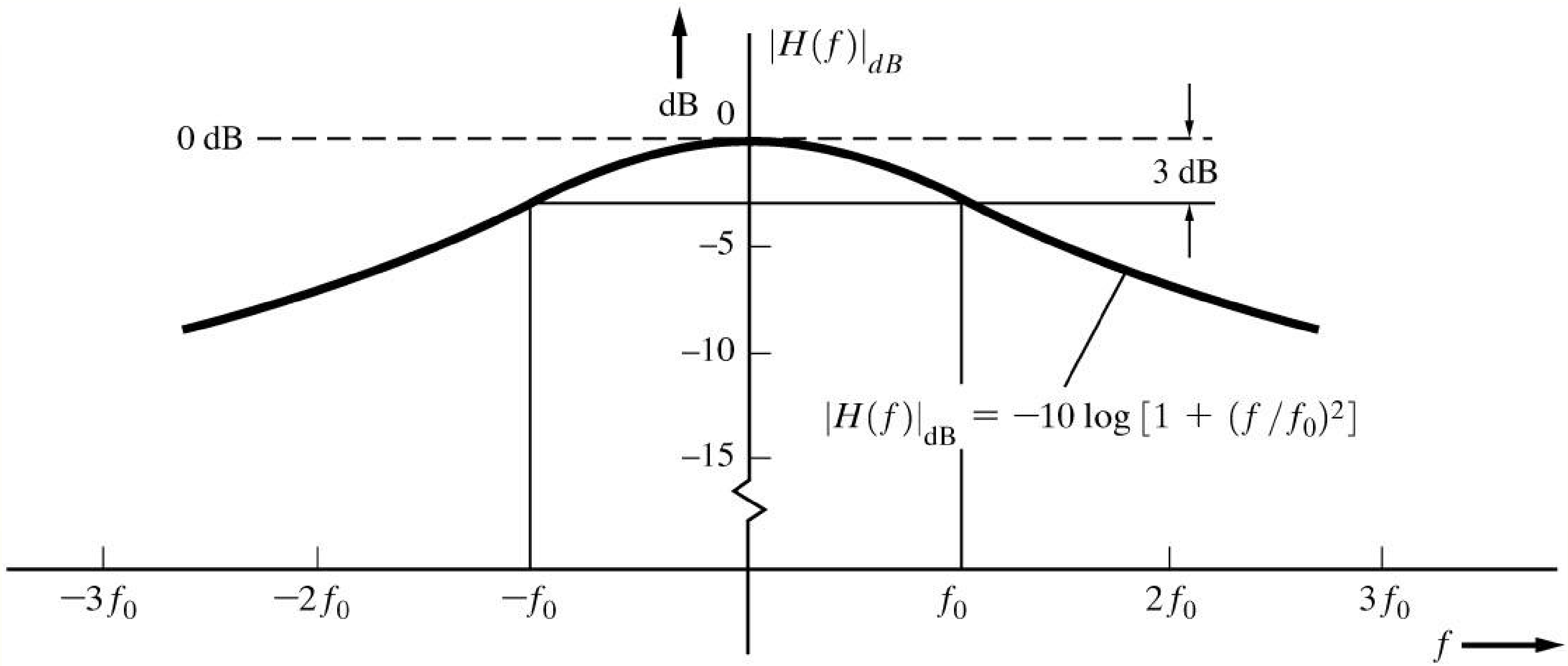


Figure 2–16 Distortion caused by an *RC* low-pass filter.



(a) Magnitude Response

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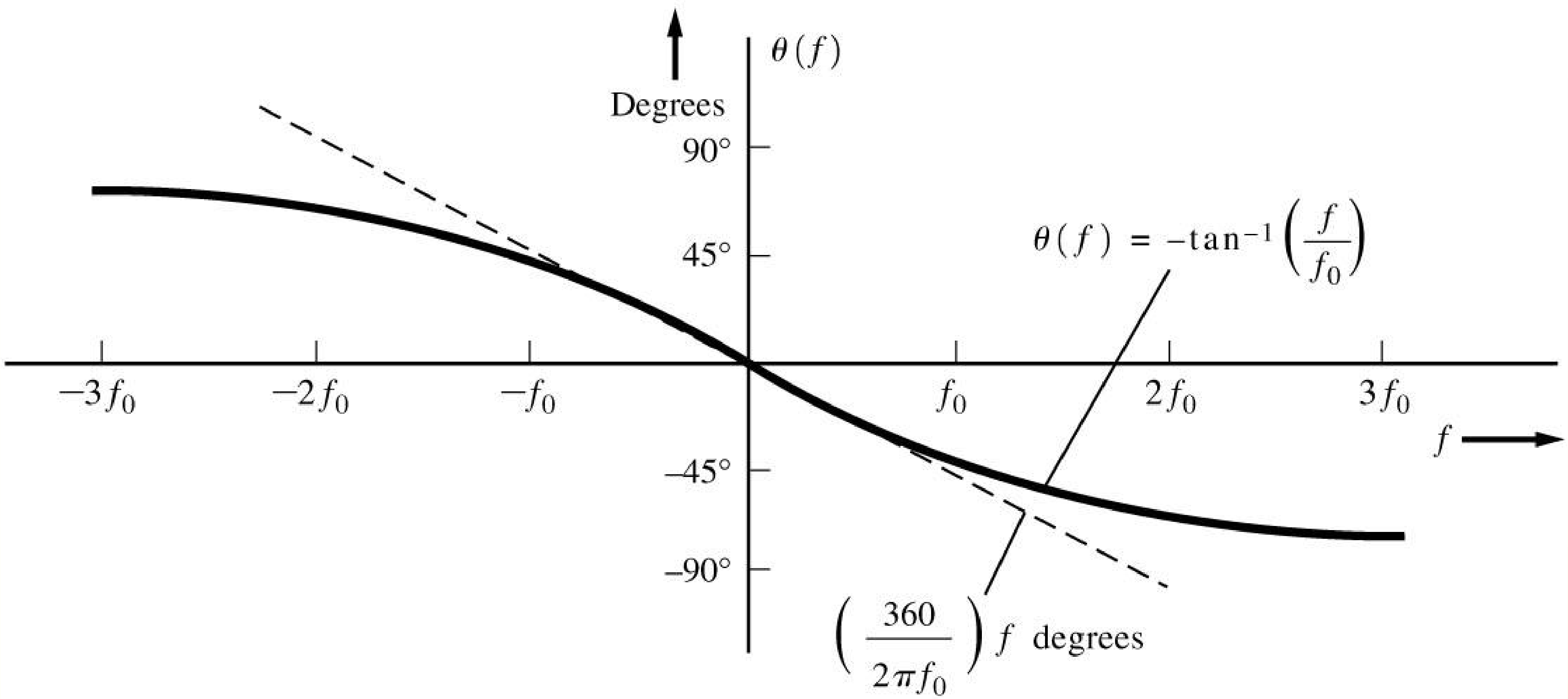
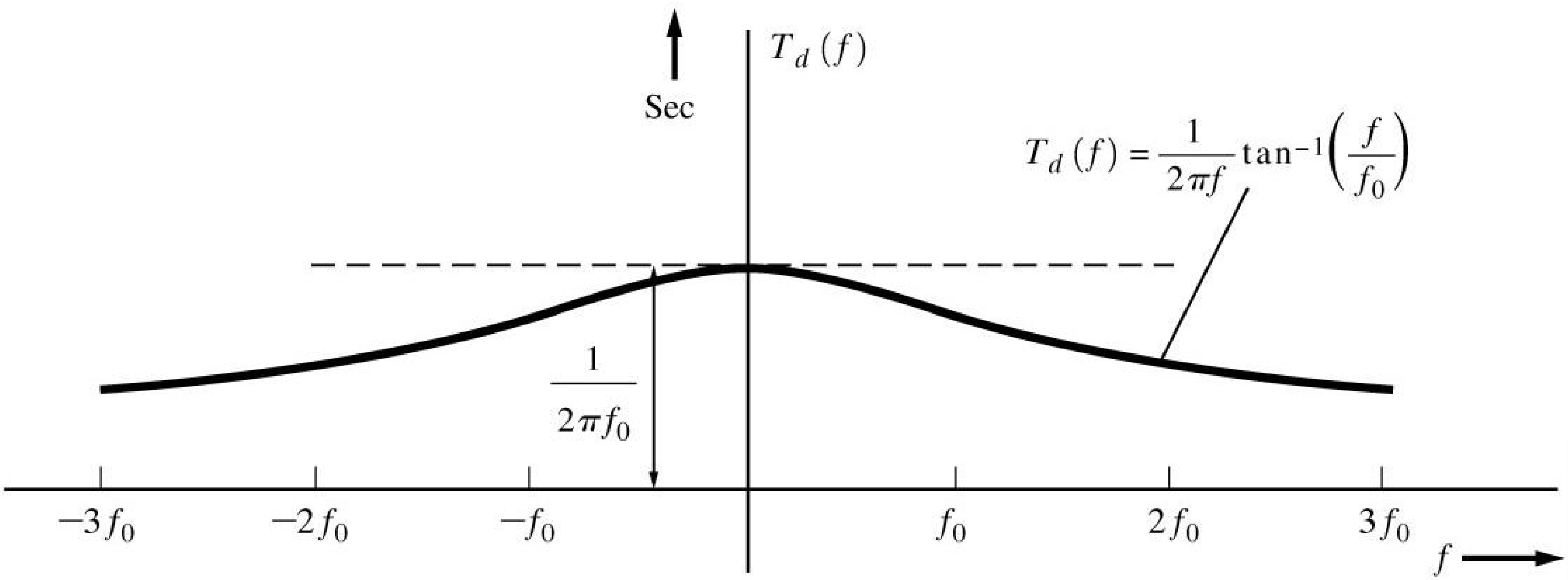
Lecture 10
Filters cont'd
and
Noise

Figure 2–16 Distortion caused by an *RC* low-pass filter.



(b) Phase Response

Figure 2–16 Distortion caused by an *RC* low-pass filter.



(c) Time Delay

Couch, *Digital and Analog Communication Systems*, Seventh Edition

A RC Filter Distortion Problem:

- Assume we want the amplitude Linearity <2%
- and the group delay variation (linearity <5%
- Find the usable bandwidth of the 1st order Butterworth filter if the 3dB bandwidth is 1 MHz

A LPF Distortion Problem:

Constraints:

$$\frac{|H(0)| - |H(f_a)|}{|H(0)|} = \varepsilon_a \le 0.02$$

2% Voltage amplitude error

$$\frac{\tau_d(0) - \tau_d(f_\phi)}{\tau_d(0)} = \varepsilon_\phi \le 0.05$$

5% delay variation

$$f_o := 10^6 \qquad \tau := \frac{1}{2\pi \cdot f_o} \qquad H(f) := \frac{1}{1 + j \cdot \frac{f}{f_o}} \qquad \tau_d(f) := \frac{1}{2\pi} \cdot \frac{f_o}{f^2 + f_o^2}$$

A LPF Distortion Problem:

Amplitude Error:

f_o

$$\varepsilon_a(f_a) := 1 - \frac{|H(f_a)|}{|H(0)|}$$

$$0.02 = 1 - \frac{1}{\sqrt{1 + \left(\frac{f_a}{f_o}\right)^2}}$$

$$f_a := f_o\sqrt{\left(\frac{1}{0.98}\right)^2 - 1}$$

$$f_a = 2.031 \times 10^5 \text{ Hz}$$

Phase Error:

$$\varepsilon_\phi(f_p) := 1 - \frac{\tau_d(f_p)}{\tau_d(0)}$$

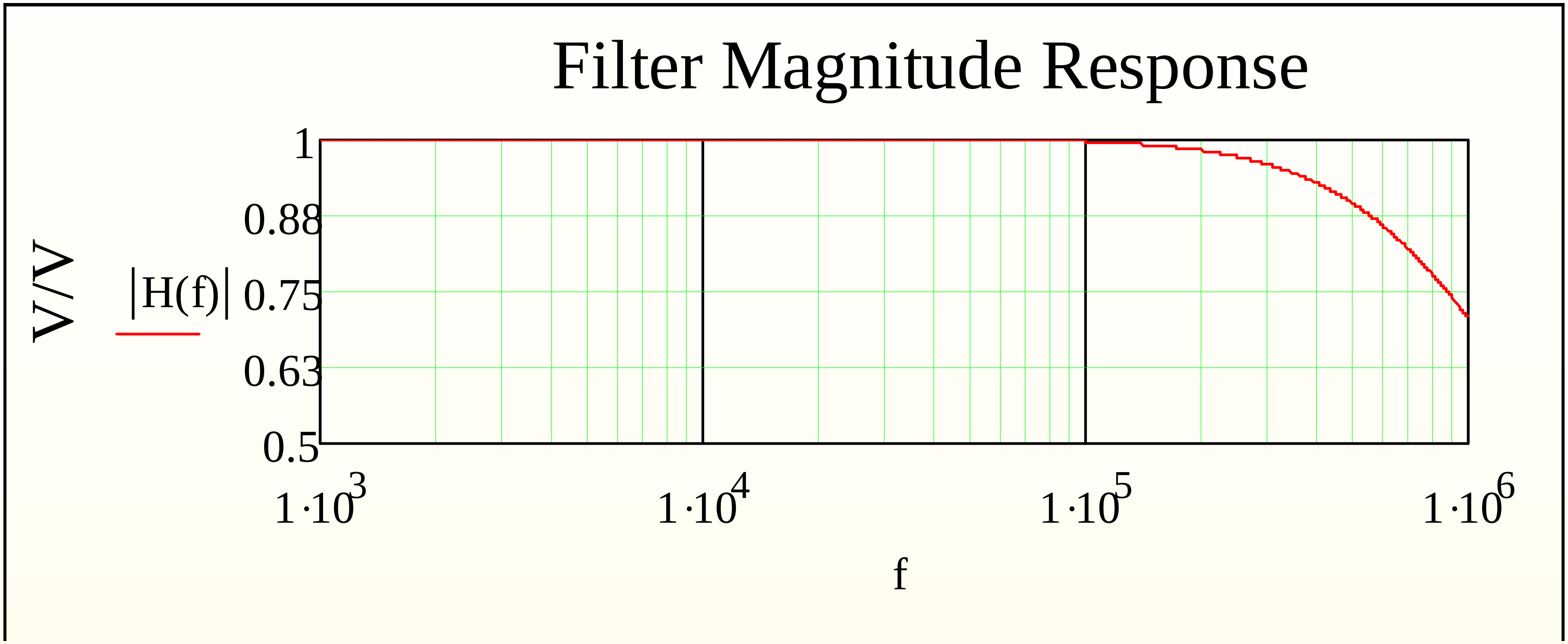
$$0.95 = \frac{f_o^2}{f_p^2 + f_o^2}$$

$$f_p := \sqrt{\frac{f_o^2}{0.95} - f_o^2}$$

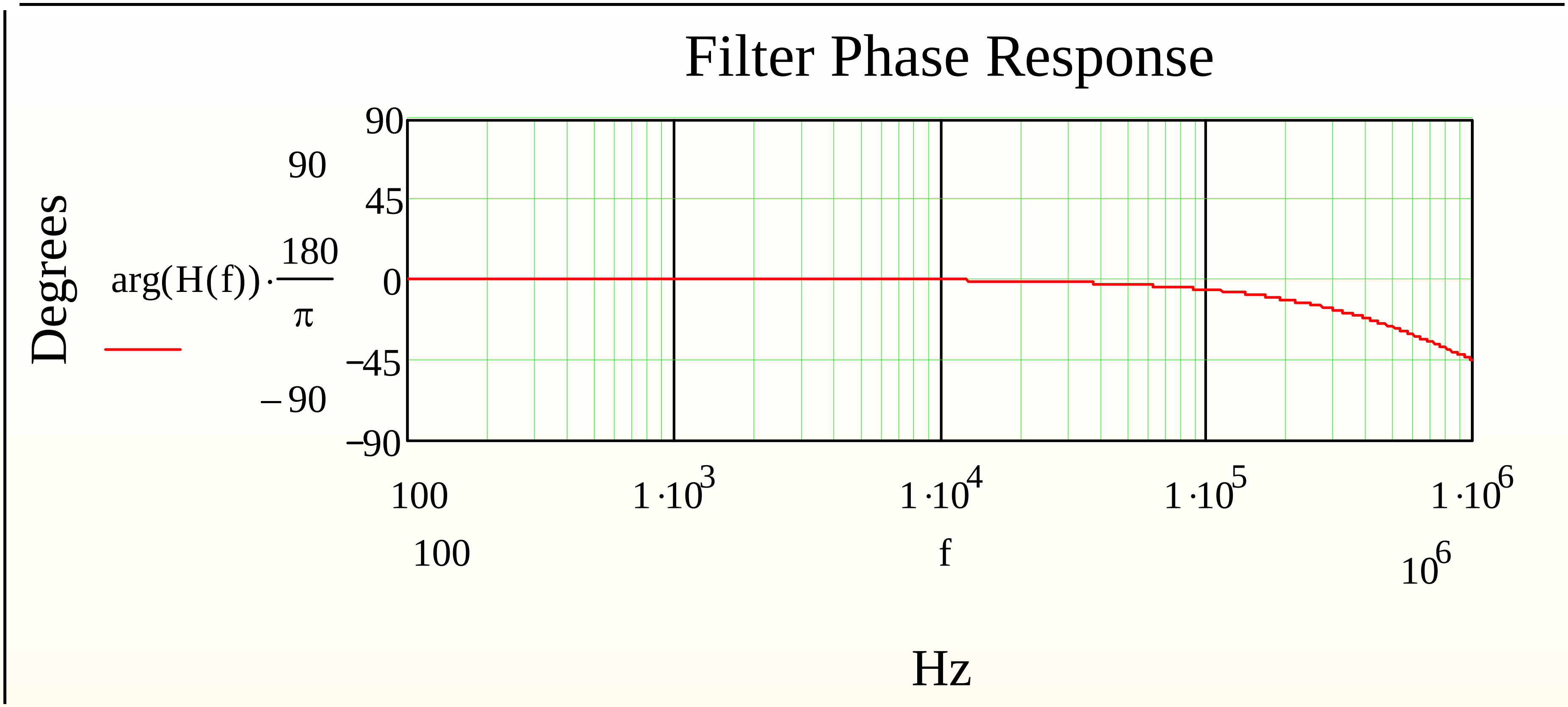
$$f_p = 2.294 \times 10^5 \text{ Hz}$$

A LPF Distortion Problem:

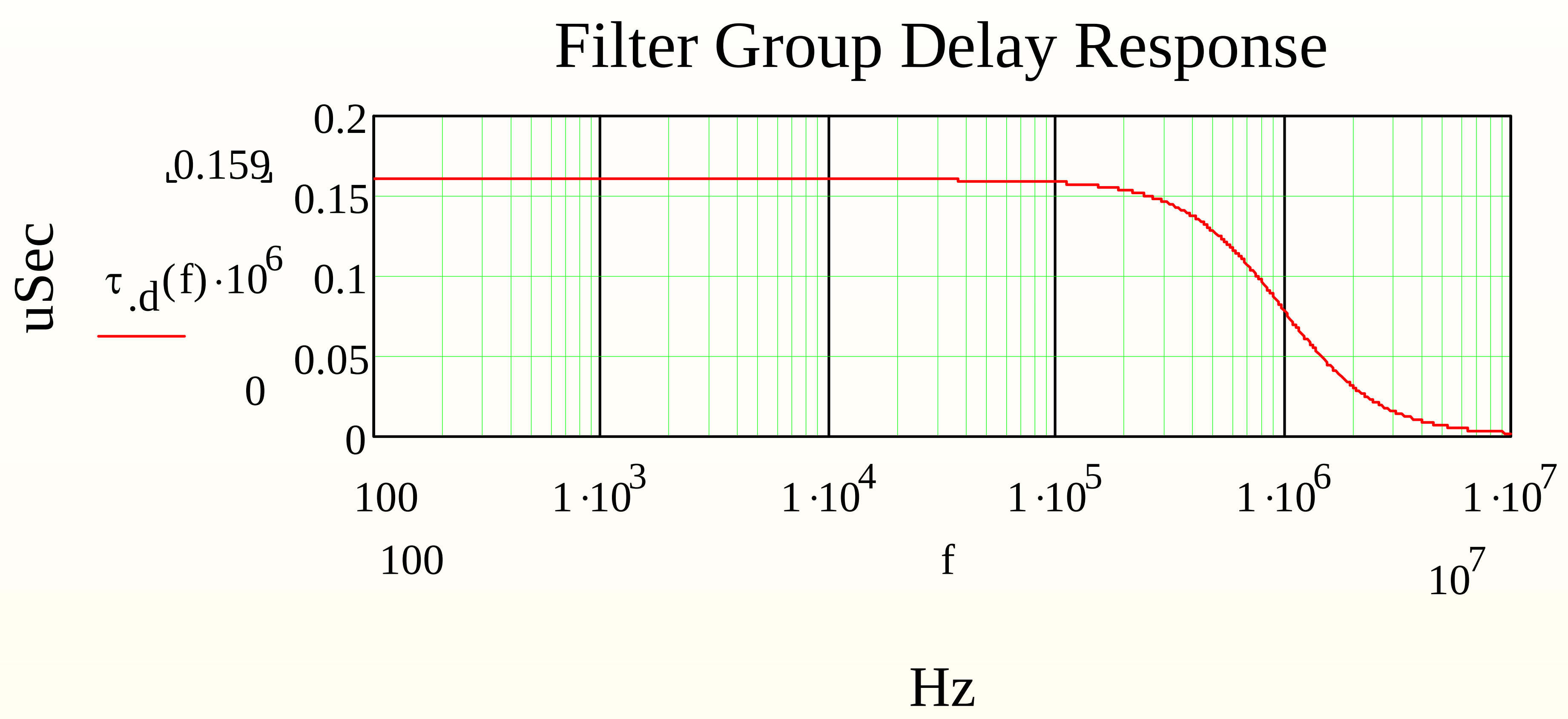
So the amplitude error will limit the usable bandwidth to 203 KHz



A LPF Distortion Problem:

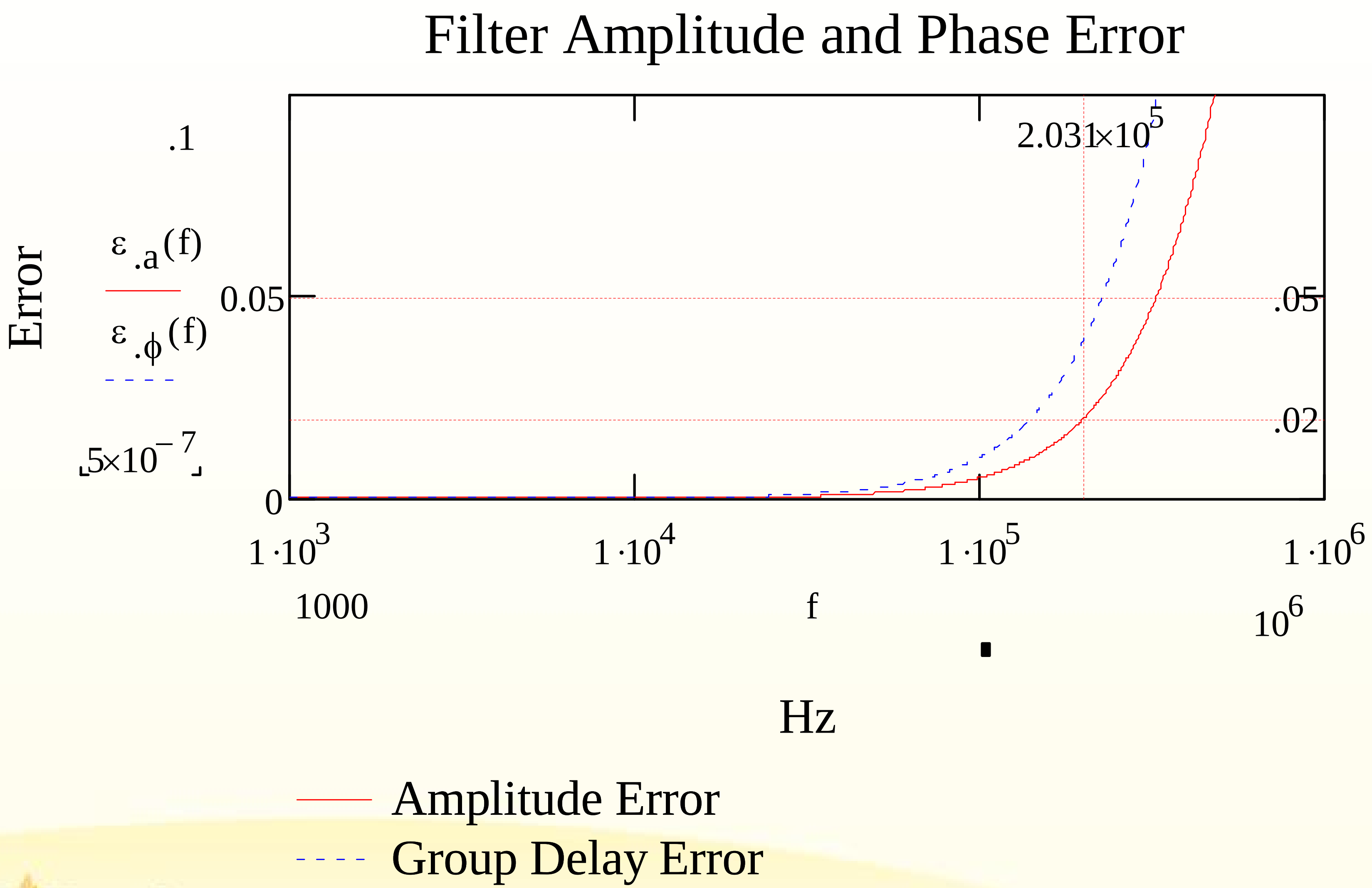


A LPF Distortion Problem:



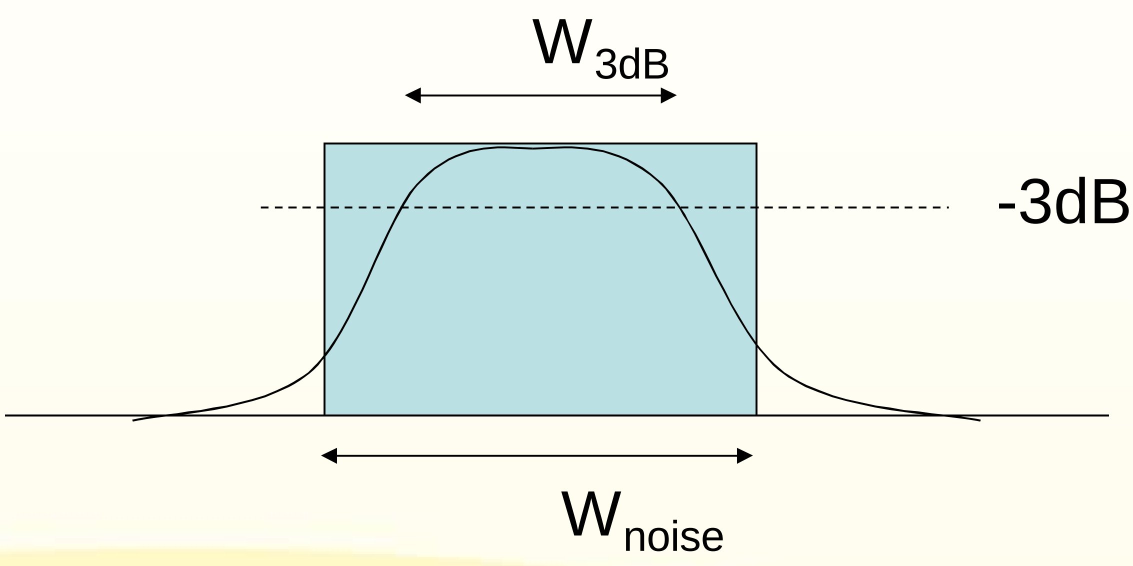
Mountains & Minds

A LPF Distortion Problem:

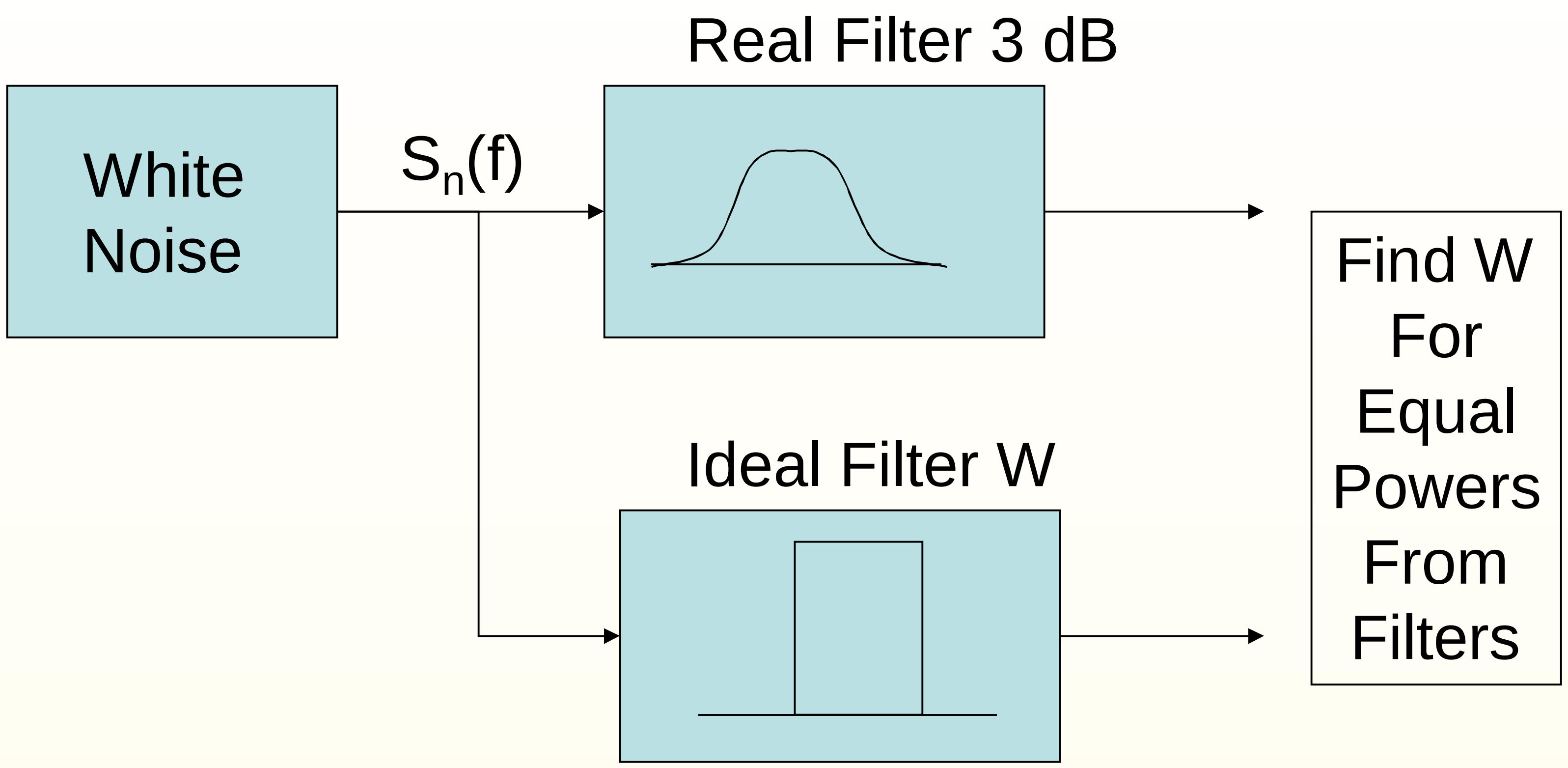


Filter Noise Equivalent Bandwidth

We often equate the -3dB bandwidth of a real Filter to the bandwidth of an ideal filter that would Pass the same noise power.



Filter Noise Equivalent Bandwidth



$S_n(f)$ is a White Noise Power Density N_o Watts/Hz

Filter Noise Equivalent Bandwidth

$$P = \int_{-\infty}^{\infty} S_n(f) \cdot \left(\left|H(f)\right|\right)^2 df = N_o \cdot \int_0^{\infty} \left(\left|H(f)\right|\right)^2 df$$

$$P = \int_{-\infty}^{\infty} S_n(f) \cdot \Pi\left(\frac{f}{2W_{eq}}\right)^2 df = \int_{-W_{eq}}^{W_{eq}} \frac{N_o}{2} df = N_o \cdot W_{eq}$$

$$W_{eq} = \int_0^{\infty} \left(\left|H(f)\right|\right)^2 df$$

Butterworth lowpass filters

$$(|H(f)|)^2 = \frac{1}{1 + \left(\frac{f}{f_o}\right)^{2\cdot n}}$$

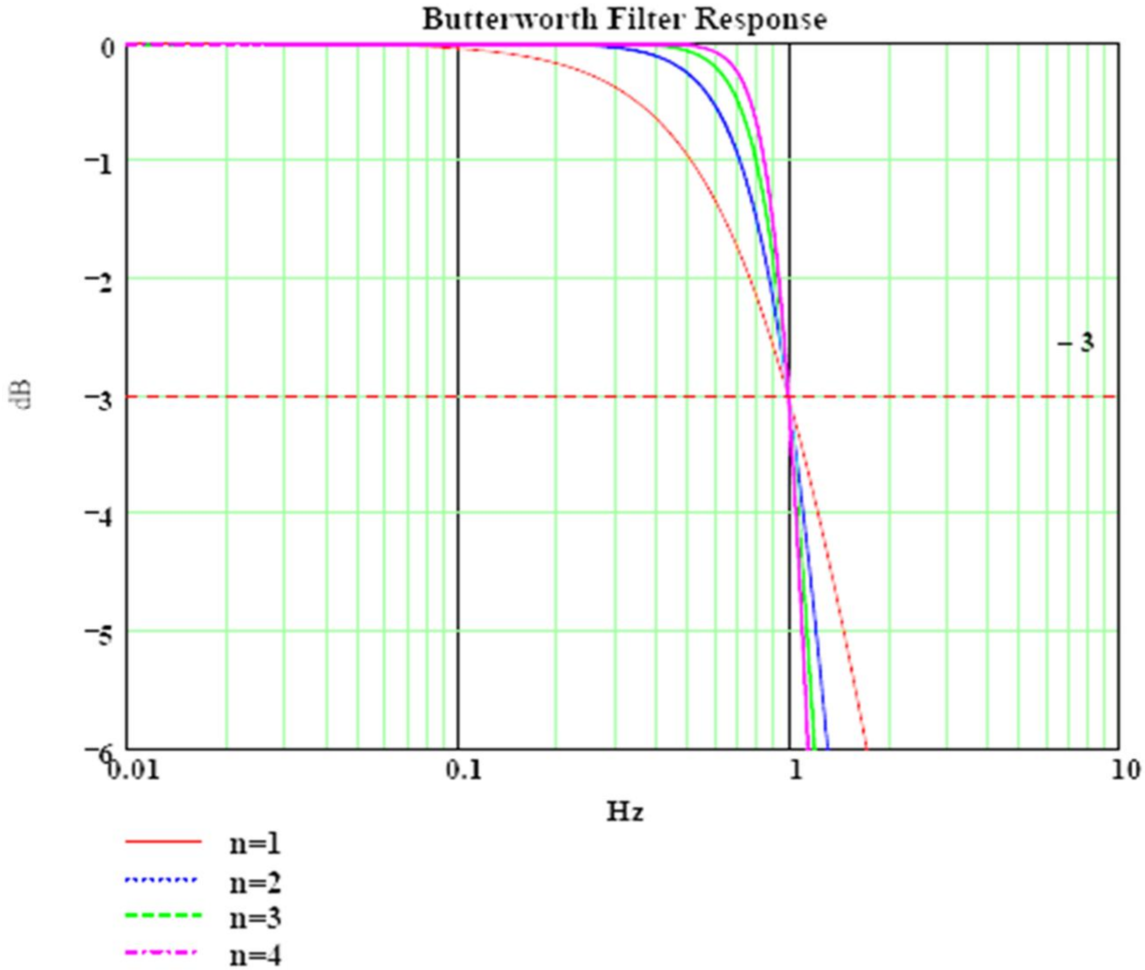
•Where n is the filter order (number of poles)

• Note that n=1 is the RC lowpass

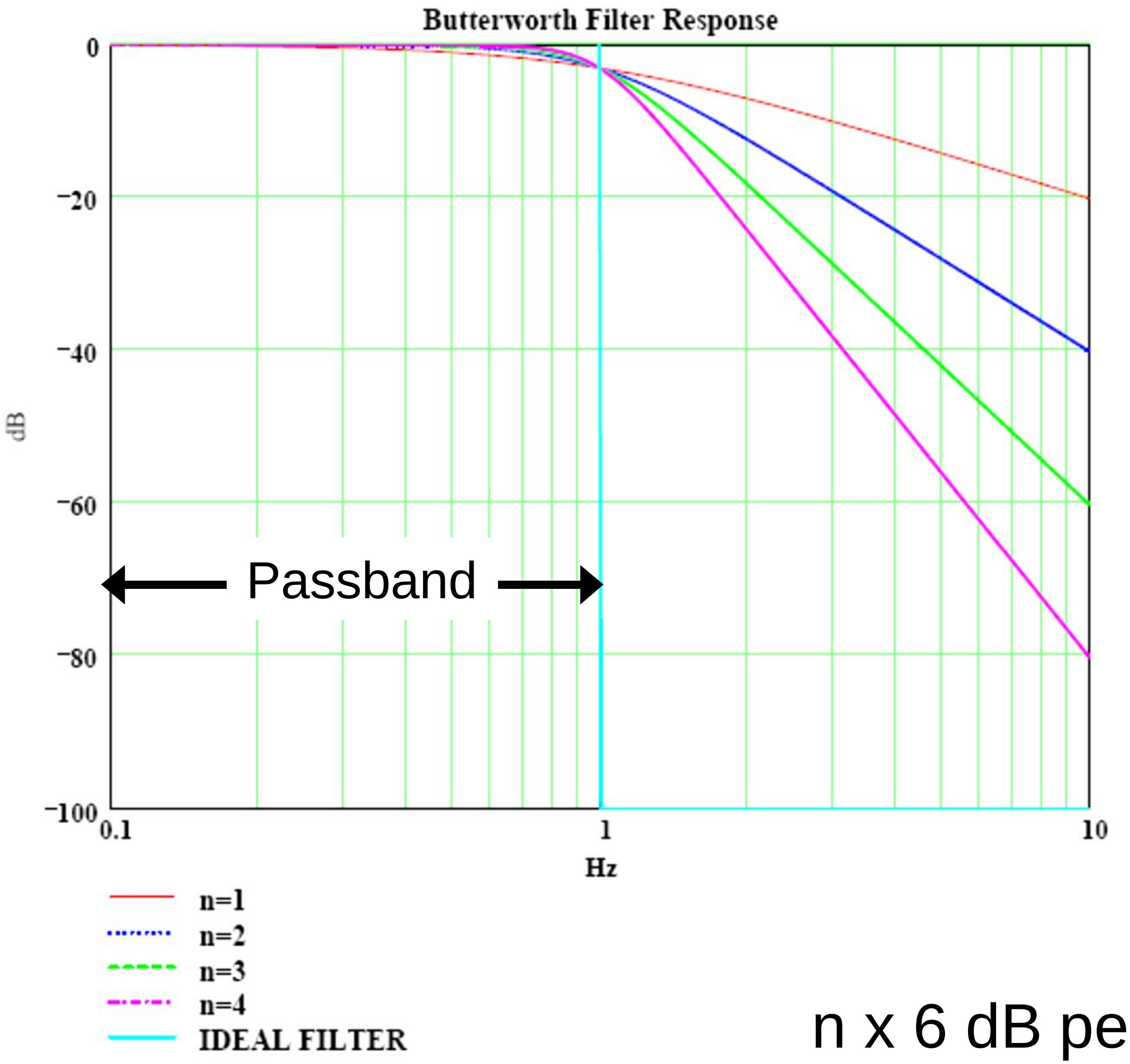
Butterworth lowpass filters

$$H(f,n) := \sqrt{\frac{1}{1+(f)^{2\cdot n}}}$$

$$H_{dB}(f,n) := 10\cdot\log\left[\frac{1}{1+(f)^{2\cdot n}}\right]$$



Butterworth lowpass filters



n x 6 dB per octave
n x 20 dB per decade

Butterworth lowpass filters

EQUIVELENT NOISE BANDWITH FOR BUTTERWORTH FILTERS

$$B = \frac{\text{Noise\_Power\_From\_Real\_Filter\_With\_3db\_Bandwidth\_of\_1}}{\text{Noise\_Power\_From\_Ideal\_Filter\_With\_Bandwidth\_of\_1}}$$

$n := 1..6$ filter orders from 1 to 6

$$B_n := \frac{\displaystyle\int_0^\infty \frac{1}{1+(f)^{2\cdot n}}\,df}{\displaystyle\int_0^\infty \Pi(f,1)^2\,df}$$

note that:

$$\int_0^\infty \frac{1}{1+(f)^2}\,df = \frac{1}{2}\cdot\pi$$

$n =$	$B_n =$
1	1.571
2	1.111
3	1.047
4	1.026
5	1.017
6	1.012

$$\frac{\pi}{2} = 1.571$$

First order filter with a 3dB bandwidth of 1 passes 57% more noise power when compared with an ideal filter with a bandwidth of 1. A 3rd order filter only passes 4.7% more noise power when compared with an ideal filter.