Chapter 5

Valuing Options / Volatility Measures

Now that the foundation regarding the basics of futures and options contracts has been set, we now move to discuss the role of volatility in futures and options markets. Volatility plays a role not only in characterizing the risk associated with a contract but also the implied price associated with an options contract. This section will demonstrate these connections throught he use of binomial trees and the Black-Scholes-Merton implied volatility measures.

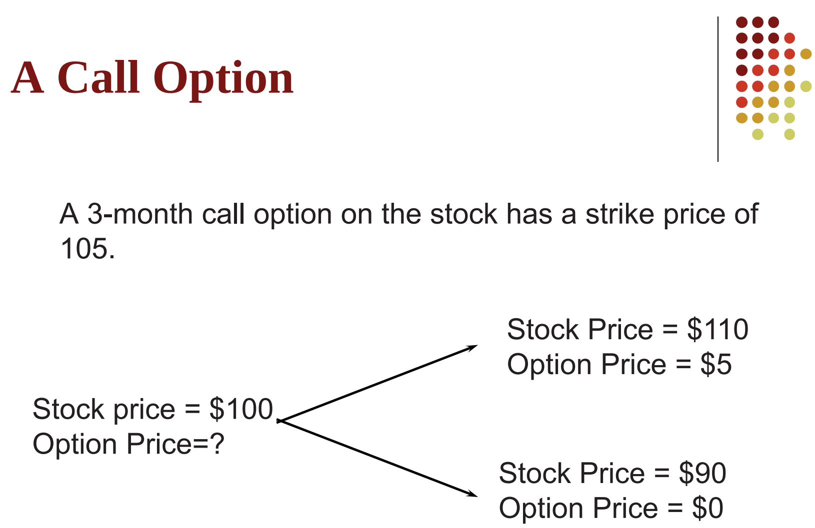
5.1 Binomial Tree

A binomial tree is one way to illustrate the pricing of an option based on possible outcomes. While the illustration here will be simple, there are more general ways to use binomial trees to derive implied prices associated with different option contracts. One important concept here is that of **risk-neutral valuation**.

5.1.1 A one-step example

Consider the following situations where

- A futures price is currently set at \$100
- In three months the price will either increase to \$110 or decrease to \$90
- We are interested in valuing a European call option to buy the stock for \$105 in three months
- This means that if the price increases the call option is valued at \$5 and for a decrease valued at \$0.
- We assume there are **no arbitrage opportunities**



These assumptions allow us to evaluate a risk-free portfolio that consists of Δ long futures contracts and a short call option. Since the portfoliios is free of risk, it must earn a return equal to the risk-free interest rate. We look to solve for the Δ such that the portfolio is risk-less. If the futures price:

1. increases to $110, then the value of the futures contract is equal to $\$110\Delta$ while the call option is worth $5.

2. decreases to $90, then the value of the futures contract is equal to $\$90\Delta$ while the call option is worth $0.

The portfolio is riskless if the value of Δ is chosen such that the two price movements yield the same returns.

$$110\Delta - 5 = 90\Delta \tag{5.1}$$

$$\Delta = 0.25 \tag{5.2}$$

Thus, a risk-less portfolio would contain 0.25 long futures shares and 1 call option. The portfolio is riskless because the returns are identical ($110*0.25-$5=$90*.025=$22.5)whether prices move up or down.

As mentioned earlier, a risk-less portfolio must earn exactly a risk-free rate of interest. Let's assume that the risk-free interest rate is 12% per annum. It follows that the followwing is the value of the portfolio today is equal to

$$22.5 \exp^{-0.12\times 3/12} = 21.83502 \tag{5.3}$$

Notice that the implied volatility in the above equation is assumed to be 12% (more on this later). Suppose the option price is denoted by f and recall the current price is $100. The value of the portfolio today can be written as

$$
\begin{aligned}
100\Delta - f &= 100 * 0.25 - f && (5.4)\\
&= 25 - f && (5.5)\\
&= 21.83502(\texttt{risk-free return rate}) && (5.6)\\
f &= 3.16498 && (5.7)
\end{aligned}
$$

Thus, the price of the option must be equal to \$3.165.

- If the value of the option were more, the portfolio would cost less than 21.83502 to set up and earn greater than a risk-free rate of return
- If the value of the option were less, the the portfolio would provide a way of borrowing money at less than the risk-free rate

5.1.2 Important assumptions

- Notice that the premium rate does not include any probabiities associated with the futures price moving up or down. This is because the probabilities of future up or down movements are already incorporated into the price of the futures contract. They do not need to be taken into account when valuing an option contract.
- **Risk-neutral valuation** is used to price options contracts and derivatives.
  - A **risk-neutral** investors do not increase the expected return they require from an investment to compensate for increased risk
  - Even though many investors are not risk-neutral (they can be risk-averse, for example), options are priced based on a risk-neutral assumption

  - This implies that additional premiums are not taken in return from taking on additional risk

- The **delta (Δ)** can be interpreted as the ratio of the change in the price of the futures option relative to the change in the futures price of the underlying stock. In our example, when the price changed from \$110 to \$90, the option price changes from \$0 to \$5, such that

$$\Delta = \frac{5-0}{110-90} = \frac{5}{20} = 0.25 \tag{5.8}$$

This simple one-step binomial tree can be generalized to include puts, multiple steps (up to 20), and american options (where options can be exercised early).

5.2 The Black-Scholes-Merton (BSM) Model

The importance of this model for use in pricing Euopean options was noted when Myron Scholes and Robert Merton won the Nobel prize in Economics in 1997 (Fischer Black died in 1995).

5.2.1 Some Preliminaries

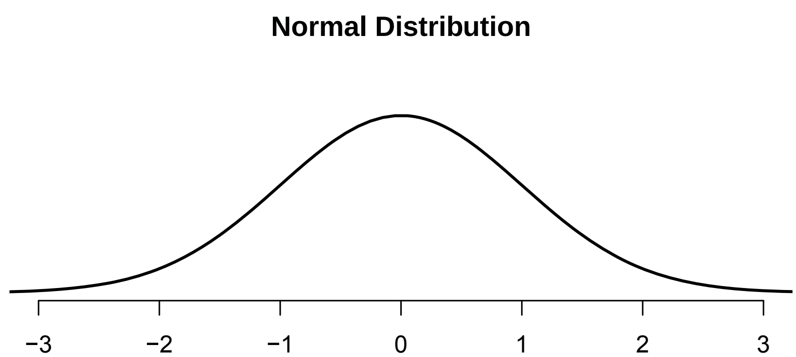
The Lognormal Distribution

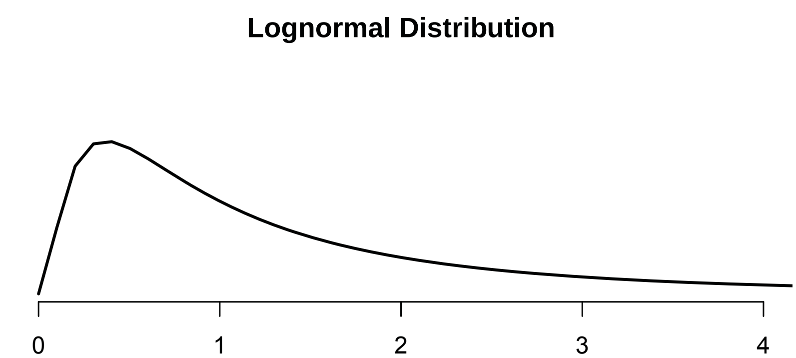
One of the fundamental assumptions of the BSM model is that the probability distribution for a non-dividend-paying stock can be distributed as follows:

$$\frac{\delta S}{S} \sim \phi(\mu\Delta t, \sigma^2\Delta t) \tag{5.9}$$

where

- ΔS is the change in the price of S in time Δt.
- $\phi(m, v)$ denotes a normal distribution with mean m and variance v.
- μ is the expected return on S
- σ is the volatility of the stock price





Notice the following differences between the normal and lognormal distributions:

In the case of prices, the assumption is often made that prices are distributed

dist	normal(μ, σ)	lognormal(μ, σ)
shape	symmetrical	positively skewed
values	neg. and pos. values	pos. values only
E(X)	μ	$e^{\mu+0.5\sigma^2}$
Var(X)	σ^2	$(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$

according to a lognormal distribution. Assume then that the price of a stock at a future time T is S_T. The assumption of lognormality implies that $log(S_T)$ is distributed according to a normal distribution.

For example, assume that the price of wheat is currently at \$6.50 per bushel (S_0) (in February). The futures price for a contract due in September (7 months from now) is currently \$6.75 per bushel, implying a annual expected return of 6.6% [$= ((6.75/6.5) - 1) * (12/7)$] with a volatility of 22.0%. To visualize what the distribution of expected prices looks like, we examine the following figure where volatility varies according to three levels (low, mid, and high).

Notice that as volatility increases a few things happen to the summary statistics:

- The tails of the distribution thicken (more extreme prices become more likely)
- The range between the 25% and 75% expands
- The mean increases due to the positive skewness of the distribution

Historical Volatility

Historical volatility is commonly reported for commodity prices. Volatility can be an important measure of how much a particular price has moved around recently. For example, assume we have stock prices S_i for $i = 1, 2, ..., n$. Historical volatility can be computed in the following way:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} \tag{5.10}$$

where $u_i = log(S_i/S_{i-1}) = log(S_i) - log(S_{i-1})$, s is the standard deviation, and $\bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$. To get to historical volatility, we use the following relationship:

$$\hat{\sigma} = \frac{s}{\tau} \tag{5.11}$$

where τ is the length of time interal in years (e.g., 1/52 for weekly intervals).

5.2.2 The BSM Pricing Formulas

The BSM formulas for the prices of Eurpean calls and puts on non-dividend-paying stocks are

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{5.12}$$

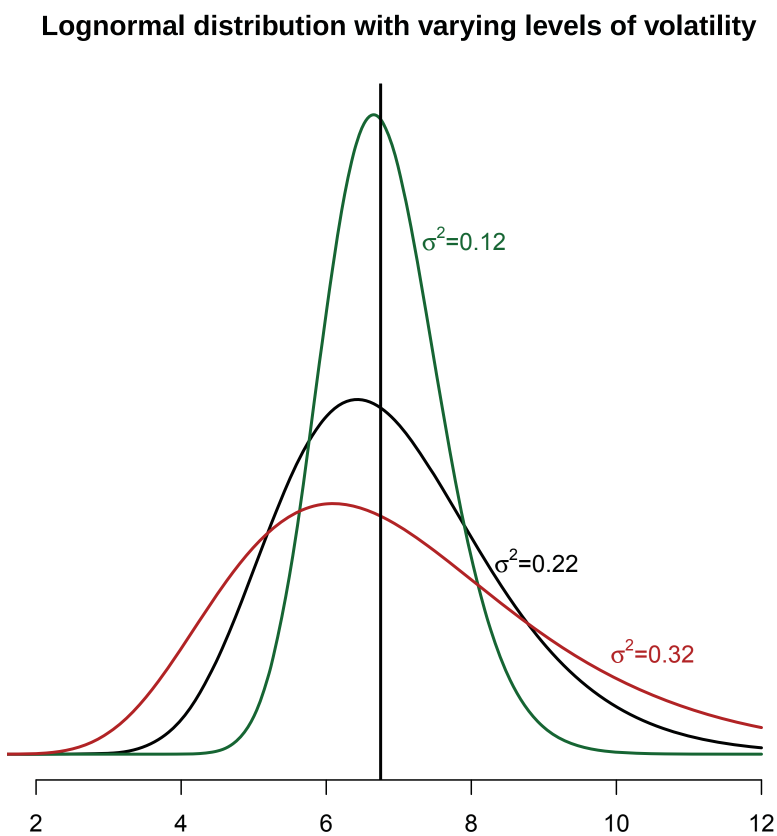
$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{5.13}$$

where

$$d_1 = \frac{log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{5.14}$$

$$d_2 = \frac{log(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{5.15}$$

$$= d_1 - \sigma\sqrt{T} \tag{5.16}$$



Volatility	25%	50%	75%	Mean	sd
0.12 (low)	6.221	6.687	7.292	6.770	0.793
0.22 (mid)	5.865	6.762	7.856	6.952	1.549
0.32 (high)	5.546	6.774	8.307	7.129	2.385