Introduction to Binomial Trees

Chapter 12

A Simple Binomial Model

- A stock price is currently \$20
- In three months it will be either \$22 or \$18

Stock price = \$20

Stock Price = \$22

Stock Price = \$18

A Call Option (Figure 12.1, page 268)

A 3-month call option on the stock has a strike price of 21.

Stock price = \$20
Option Price=?

Stock Price = \$22
Option Price = \$1

Stock Price = \$18
Option Price = \$0

Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares
  short 1 call option

$22\Delta - 1$

18Δ

- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio
(Risk-Free Rate is 12%)

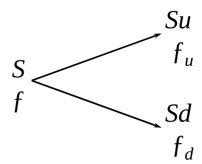
- The riskless portfolio is:
  long 0.25 shares
  short 1 call option
- The value of the portfolio in 3 months is
  $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is
  $4.5e^{-0.12\times0.25} = 4.3670$

Valuing the Option

- The portfolio that is
  long 0.25 shares
  short 1 option
  is worth 4.367
- The value of the shares is
  5.000 (= 0.25×20)
- The value of the option is therefore
  0.633 (= 5.000 – 4.367)

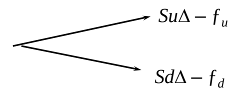
Generalization (Figure 12.2, page 269)

A derivative lasts for time T and is dependent on a stock



Generalization (continued)

- Consider the portfolio that is long Δ shares and short 1 derivative



- The portfolio is riskless when $Su\Delta - f_u = Sd\,\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{Su - Sd}$$

Generalization (continued)

- Value of the portfolio at time T is
  $Su\,\Delta - f_u$
- Value of the portfolio today is
  $(Su\,\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S\,\Delta - f$
- Hence
  $f = S\,\Delta - (Su\,\Delta - f_u)e^{-rT}$

Generalization (continued)

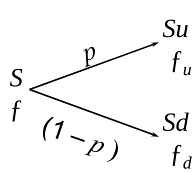
- Substituting for Δ we obtain
  $f = [\,p\,f_u + (1 - p\,)f_d\,]e^{-rT}$

where

$$p = \frac{e^{rT} - d}{u - d}$$

Risk-Neutral Valuation

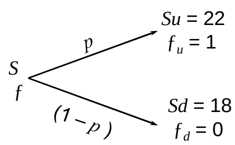
- $f = [\,p\,f_u + (1 - p\,)f_d\,]e^{-rT}$
- The variables p and $(1 - p\,)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



Irrelevance of Stock's Expected Return

When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant

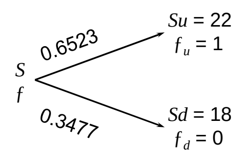
Original Example Revisited



- Since p is a risk-neutral probability
  $20e^{0.12 \times 0.25} = 22p + 18(1 - p)$; $p = 0.6523$
- Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option Using Risk-Neutral Valuation



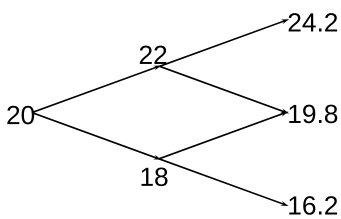
The value of the option is

$e^{-0.12\times0.25}$ [0.6523×1 + 0.3477×0]

= 0.633

A Two-Step Example

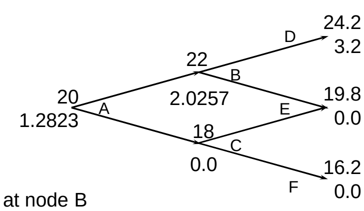
Figure 12.3, page 274



- Each time step is 3 months
- K=21, r =12%

Valuing a Call Option

Figure 12.4, page 274



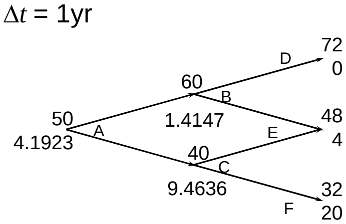
- Value at node B
  $= e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$
- Value at node A
  $= e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0)$
  = 1.2823

A Put Option Example; K=52

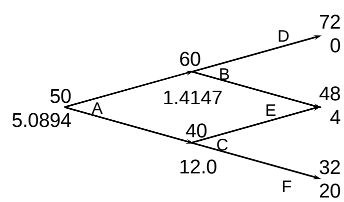
Figure 12.7, page 277

K = 52, Δt = 1yr

r = 5%



What Happens When an Option is American (Figure 12.8, page 278)



Delta

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node

Choosing *u* and *d*

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

The Probability of an Up Move

$$p = \frac{a-d}{u-d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk-free rate

$a = 1$ for a futures contract

Increasing the Time Steps

- In practice at least 30 time steps are necessary to give good option values
- DerivaGem allows up to 500 time steps to be used