Valuing Stock Options: The Black-Scholes-Merton Model

Chapter 13

The Black-Scholes-Merton Random Walk Assumption

- Consider a stock whose price is S
- In a short period of time of length Δt the return on the stock ($\Delta S/S$) is assumed to be normal with mean $\mu\Delta t$ and standard deviation

$$\sigma\sqrt{\Delta t}$$

- μ is expected return and σ is volatility

The Lognormal Property

- These assumptions imply ln S_T is normally distributed with mean:

$$\ln S_0 + (\mu - \sigma^2/2)T$$

and standard deviation:

$$\sigma\sqrt{T}$$

- Because the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Property continued

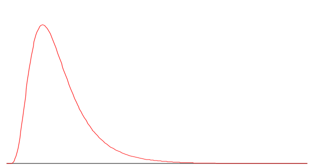
$$\ln S_T \approx \phi\left[\ln S_0 + (\mu - \sigma^2/2)T, \sigma^2 T\right]$$

or

$$\ln \frac{S_T}{S_0} \approx \phi\left[(\mu - \sigma^2/2)T, \sigma^2 T\right]$$

where $\phi\,[m,v]$ is a normal distribution with mean m and variance v

The Lognormal Distribution



$$E(S_T) = S_0\, e^{\mu T}$$

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

The Expected Return

- The expected value of the stock price is $S_0 e^{\mu T}$
- The return in a short period Δt *is* $\mu\Delta t$
- But the expected return on the stock with continuous compounding is $\mu - \sigma^2/2$
- This reflects the difference between arithmetic and geometric means

Mutual Fund Returns (See Business Snapshot 13.1 on page 294)

- Suppose that returns in successive years are 15%, 20%, 30%, -20% and 25%
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4%

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in time Δt is $\sigma\sqrt{\Delta t}$
- If a stock price is \$50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Nature of Volatility

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued

Estimating Volatility from Historical Data (page 295-298)

1. Take observations $S_0, S_1, \ldots, S_n$ on the variable at end of each trading day
2. Define the continuously compounded daily return as:
$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$
3. Calculate the standard deviation, s, of the u_i´s
4. The historical volatility per year estimate is: $s\times\sqrt{252}$

Estimating Volatility from Historical Data continued

- More generally, if observations are every τ years (τ might equal 1/252, 1/52 or 1/12), then the historical volatility per year estimate is
$$\frac{s}{\sqrt{\tau}}$$

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate

The Black-Scholes Formulas

(See page 299-300)

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$
$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $d_1 = \dfrac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The $N(x)$ Function

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x
- See tables at the end of the book

Properties of Black-Scholes Formula

- As S_0 becomes very large c tends to $S_0 - Ke^{-rT}$ and p tends to zero
- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S_0$

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- This is consistent with the risk-neutral valuation principle

Applying Risk-Neutral Valuation

1. Assume that the expected return from an asset is the risk-free rate
2. Calculate the expected payoff from the derivative
3. Discount at the risk-free rate

Valuing a Forward Contract with Risk-Neutral Valuation

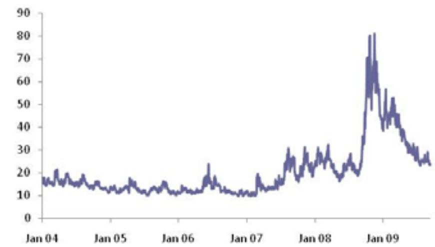
- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $S_0e^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[S_0e^{rT} - K] = S_0 - Ke^{-rT}$$

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- The is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The VIX Index of S&P 500 Implied Volatility; Jan. 2004 to Sept. 2009



Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into the Black-Scholes-Merton formula
- Only dividends with ex-dividend dates during life of option should be included
- The "dividend" should be the expected reduction in the stock price on the ex-dividend date

American Calls

- An American call on a non-dividend-paying stock should never be exercised early
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date

Black's Approximation for Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:

1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date