Linear Programming

AGEC 467

Fall 2010

The Objective

Our general problem is to optimize a linear function (which can be written several ways)

$$c_1x_1 + \ldots + c_nx_n \qquad \sum_{i=1}^{n} c_ix_i$$

$$\mathbf{c}^T \cdot \vec{x} \;\text{ where }\; \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The Constraints

Our optimal solution is subject to a set of linear inequality constraints, often with with non-negativity constraints as well. Of course, non-negativity constraints are simply a special case of an inequality constraint.

$$\begin{array}{llll} a_{11}x_1 & + \ldots & + a_{1n}x_n & \leq b_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + \ldots & + a_{mn}x_n & \leq b_m \end{array}$$
$$x_1 \ldots x_n \geq 0$$

The General Problem

Defining **A** as $\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ we can rewrite this problem using vector notation. Although this is not strictly necessary, it does have some advantages, particularly as dimensionality increases.

$$\max \mathbf{c}^T \cdot \mathbf{x} \;\; s.t. \quad \begin{aligned} \mathbf{A}\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Geometry of the Problem

For a well-defined maximization, we need a constraint set defined by lower halfspaces, as show below at left. The corresponding upper halfspace-defined constraint set for a minimization is shown below at right.

Maximization constraint set

Minimization constraint set

There are numerous ways to solve linear programming problems. We will talk about 4 main techniques.

1. Graphical
2. Substitution
3. Slack and Surplus Variables
4. Simplex Algorithm

Graphical Solutions

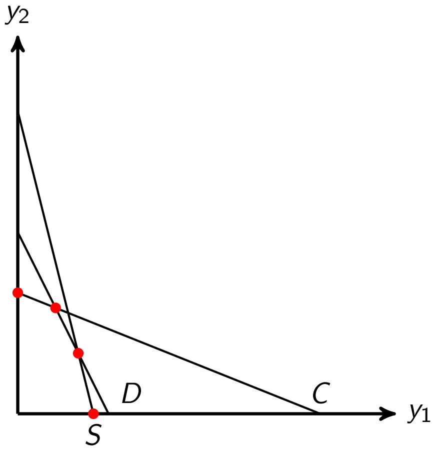
Graphical solutions are just that–draw a picture and figure out what the optimal solution is. The limitation on graphical solutions is that for high-dimension problems. However, for low-dimension problems, graphical solutions can be cheap and easy.

Example: Manufacturer's Problem

$$\begin{aligned} \max 24y_1 + 8y_2 \qquad s.t. \quad & 2y_1 + 5y_2 \leq 40 && \text{Chopping} \\ & 4y_1 + y_2 \leq 20 && \text{Slicing} \\ & 10y_1 + 5y_2 \leq 60 && \text{Dicing} \\ & y_1, y_2 \geq 0 \end{aligned}$$

Graphical Solution

Draw a picture of the manufacturer's problem.



Identifying Vertices

Visual inspection informs us that none of the constraints are collinear with the objective function. Therefore we can find the corners defined by the constraints and be assured that the optimum will occur at a cusp of the constraint set. Those corners are:

$$(0, 8) \quad (\frac{5}{2}, 7) \quad (4, 4) \quad (5, 0)$$

so we can simply evaluate the profit function at each point to determine the optimal output combination.

Evaluating Candidate Solutions

$$\pi(0,8) = 64 \quad \pi(\frac{5}{2},7) = 116 \quad \pi(4,4) = 128 \quad \pi(5,0) = 120$$

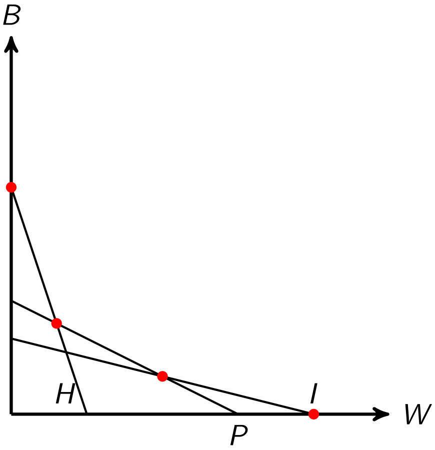
We find that (4,4) dominates all of these possibilities with profits of 128. The slicing and dicing constraints bind at this solution, while the chopping constraint holds as an inequality.

Minimization Example

Consider the farmer's problem. She seeks to minimize her costs from the production of two crops (W and B) when growing crops requires three (limited) activities: planting, irrigating, and harvesting.

$$\begin{aligned} \min 12W + 42B \qquad s.t. \quad & W + 2B \geq 3 && \text{Planting} \\ & W + 4B \geq 4 && \text{Irrigating} \\ & 3W + B \geq 3 && \text{Harvesting} \\ & W, B \geq 0 \end{aligned}$$

Minimization Example



Minimization Example

Since all of the constraints are linear, we can easily calculate the locations of all of the intersections of the constraints in the non-negative orthant (W, B space).

$$(4,0) \quad (2,\frac{1}{2}) \quad (\frac{3}{5},\frac{6}{5}) \quad (0,3)$$

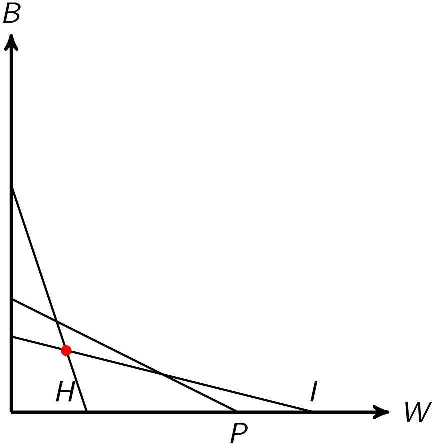
We can then calculate the value of the cost function $(12W + 42B)$ at each of these points.

$$C(4,0) = 48 \quad C(2,\frac{1}{2}) = 45 \quad C(\frac{3}{5},\frac{6}{5}) = 57.6 \quad C(0,3) = 126$$

It is straightforward to see that $(2,\frac{1}{2})$ is the optimal combination.

Minimization Example

What about C($\frac{8}{11}$,$\frac{9}{11}$)=43 $\frac{1}{11}$, which represents the intersection of the I and H constraints?



Although the cost is in fact lower than our previous optimum, we find that this output vector does not satisfy the P constraint. This type of solution is called *inadmissable*. After experience with concave programs, this is equivalent to violating the constraint set.

LP with a Lagrangian

For comparison, try to solve the manufacturer's problem by concave programming.

$$\begin{array}{llll} \max 24y_1 + 8y_2 & s.t. & 2y_1 + 5y_2 \leq 40 & \text{Chopping} \\ & & 4y_1 + y_2 \leq 20 & \text{Slicing} \\ & & 10y_1 + 5y_2 \leq 60 & \text{Dicing} \\ & & y_1, y_2 \geq 0 & \end{array}$$

$$L = 24y_1 + 8y_2 + \lambda(40 - 2y_1 - 5y_2) + \mu(20 - 4y_1 - y_2) + \nu(60 - 10y_1 - 5y_2)$$

LP with a Lagrangian

K-T FONC

$$
\begin{array}{ll}
24 - 2\lambda - 4\mu - 10\nu \leq 0 & y_1\left(24 - 2\lambda - 4\mu - 10\nu\right) = 0 \\
8 - 5\lambda - \mu - 5\nu \leq 0 & y_2\left(8 - 5\lambda - \mu - 5\nu\right) = 0 \\
40 - 2y_1 - 5y_2 \geq 0 & \lambda\left(40 - 2y_1 - 5y_2\right) = 0 \\
20 - 4y_1 - y_2 \geq 0 & \mu\left(20 - 4y_1 - y_2\right) = 0 \\
60 - 10y_1 - 5y_2 \geq 0 & \nu\left(60 - 10y_1 - 5y_2\right) = 0
\end{array}
$$

LP with a Lagrangian

This yields a solution.

$$y_1 = 4$$

$$y_2 = 4$$

$$\lambda = 0$$

$$\mu = 4$$

$$\nu = \frac{4}{5}$$

So the concave program works here, but it may be less convenient than some of the other solution methods we'll discuss.

Substitution Method

Another option is to use algebra techniques to solve the system by substitution. This is possible when graphical solutions are not. Recall the manufacturer's problem.

Now express y_2 as a function of y_1 in the objective function and substitute into the constraints. Note that you could make the substitution the other way with equal ease.

$\pi = 24y_1 + 8y_2 \rightarrow \frac{\pi}{8} - 3y_1 = y_2$
$\frac{5\pi}{8} - 13y_1 \leq 40$
$\frac{\pi}{8} + y_1 \leq 20$
$\frac{5\pi}{8} - 5y_1 \leq 60$

Substitution Method

All of these expression must hold at a solution. Some will be equalities, others inequalities. Rearrange the second expression and substitute it into the third to solve for the value of profit. This tells us that the optimal profit must be less than or equal to 128. Armed with this information we can substitute back into the constraints and simplify.

$y_1 \geq \frac{40}{13}$
$y_1 \leq 4$
$y_1 \geq 4$

The last two conditions lead us to believe that $y_1 = 4$, which satisfies the first constraint. Substituting back into the objective function we quickly discover that $y_2 = 4$ and $\pi = 128$. Fortunately, this the same solution we obtained graphically.

Slack Variables

Another solution method for a maximization problem is to introduce *slack variables* to each of the constraints. These variables will measure the amount of slack in each of the constraints. In the event of a minimization problem we introduce *surplus variables*, since the geometry of the problem dictates that the feasible set is above the boundary.
Introducing these variables will permit us to solve a concave program with equalities.

Slack Variables in a Maximization Problem

$$40 - 2y_1 - 5y_2 + \sigma_1 = 0$$
$$20 - 4y_1 - y_2 + \sigma_2 = 0$$
$$60 - 10y_1 - 5y_2 + \sigma_3 = 0$$

Now we can make an appropriate partition and choose which three unknowns to include in the solution. Binding constraints will have slack variables that equal zero. In this case we know to set σ_2 and σ_3 to zero.

Surplus Variables in a Minimization Problem

$W + 2B - \tau_1 = 3$
W + 4B - $\tau_2 = 4$
3W + B - $\tau_3 = 3$

Now we can make an appropriate partition and choose which three unknowns to include in the solution. One way to systematically solve using slack or surplus variables is the *simplex algorithm*.

Simplex Algorithm

Another possible solution method is the *simplex algorithm*. Learning this method gives us an additional solution method for LP problems, but also some insight into how simplex methods are used for econometric problems.
Examples of applications of simplex methods are for multivariate absolute deviation estimators (like quantile or median regressions), and some types of nonlinear optimizing algorithms (like EM).

Steps in simplex method (max)

1. set up matrix form augmented with added slack variables, constant vector, and subtracting objective function coefficients in bottom row
2. pivot by
   1. (column) selecting largest negative value in objective function
   2. (row) smallest displacement (ratio $\frac{\text{constant}}{\text{coefficients}}$) by row
   3. perform row operations on <u>all</u> rows, including objective function
3. repeat (2) as long as negative elements remain in bottom row
4. read solution from RH column, including objective function value

Setting up a simplex array

Revisit the manufacturer's problem. We'll need a row for each constraint (3), plus a bottom row. The number of columns is 6—two choice variables, three slack variables, and a coefficient column.

Constraints

First plug in the values of the matrix form in the first two columns.

2	5		
4	1		
10	5		

Slack Variables

Now add slack variables to array.

$$\begin{array}{ccccc|c}
2 & 5 & 1 & 0 & 0 & \\
4 & 1 & 0 & 1 & 0 & \\
10 & 5 & 0 & 0 & 1 & \\
\hline
\end{array}$$

Coefficients

The next step is to include the coefficients in the far right-hand column.

$$\begin{array}{ccccc|c}
2 & 5 & 1 & 0 & 0 & 40 \\
4 & 1 & 0 & 1 & 0 & 20 \\
10 & 5 & 0 & 0 & 1 & 60 \\
\hline
\end{array}$$

Objective Function

Now we need to fill in the bottom row. Start by including the negative of the objective function coefficients in the bottom row.

$$\begin{array}{ccccc|c}
2 & 5 & 1 & 0 & 0 & 40 \\
4 & 1 & 0 & 1 & 0 & 20 \\
10 & 5 & 0 & 0 & 1 & 60 \\
\hline
-24 & -8 & & & &
\end{array}$$

Identifying the Origin

Complete the array by entering 0 for each of the slack variables in the bottom row, and a zero in the bottom right corner (the value of the objective function).

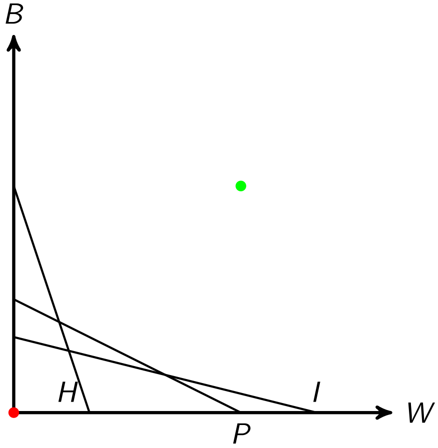
$$\begin{array}{ccccc|c}
2 & 5 & 1 & 0 & 0 & 40 \\
4 & 1 & 0 & 1 & 0 & 20 \\
10 & 5 & 0 & 0 & 1 & 60 \\
\hline
-24 & -8 & 0 & 0 & 0 & 0
\end{array}$$

Steps in simplex method (min)

1. set up matrix form augmented by *subtracting* slack variables, adding M times column sums, constant, and subtracting objective function coefficients
2. pivot by
   1. (column) selecting largest *positive* value in objective function
   2. (row) smallest displacement (ratio $\frac{\text{constant}}{\text{coefficients}}$) by row
   3. perform row operations on all rows
3. repeat (2) as long as positive elements remain in bottom row
4. read solution from RH column, including objective function value

Geometric Motivation for Minimization Simplex

The origin is not in the feasible set. We need a starting point in the feasible set, so we pick an arbitrary one.



Example: Farmer's Problem

The farmer seeks to minimize the costs of growing either wheat W or barley B subject to constraints on planting P, irrigating I, and harvesting H.

$$
\min 12W + 42B \qquad s.t. \quad \begin{aligned} W + 2B &\geq 3 \\ W + 4B &\geq 4 \\ 3W + B &\geq 3 \\ W, B &\geq 0 \end{aligned}
$$

Set up the simplex array.

$$
\begin{array}{ccccc|c}
1 & 2 & -1 & 0 & 0 & 3 \\
1 & 4 & 0 & -1 & 0 & 4 \\
3 & 1 & 0 & 0 & -1 & 3 \\
\hline
5M-12 & 7M-42 & -M & -M & -M & 10M
\end{array}
$$

Example: Farmer's Problem

Note that the bottom row is $-f + \text{row sum} \cdot M$.
Now pick the biggest element of bottom row (remembering that M is an arbitrarily large number) and pivot on the row with the smallest displacement (smallest ratio of constant and coefficient).
In this case the optimal pivot is column 2, row 2 ($1 < 1.5 < 3$).
STEP 1a: divide row 2 by 4

$$
\begin{array}{ccccc|c}
1 & 2 & -1 & 0 & 0 & 3 \\
\frac{1}{4} & 1 & 0 & -\frac{1}{4} & 0 & 1 \\
3 & 1 & 0 & 0 & -1 & 3 \\
\hline
5M-12 & 7M-42 & -M & -M & -M & 10M
\end{array}
$$

Example: Farmer's Problem

STEP 1b: subtract 2x row 2 from row 1, 1x row 2 from row 3, $\frac{7M-42}{4}$ times row 2 from row 4

$$\begin{array}{ccccc|c}
\frac{1}{2} & 0 & -1 & -\frac{1}{2} & 0 & 1 \\
\frac{1}{4} & 1 & 0 & -\frac{1}{4} & 0 & 1 \\
\frac{11}{4} & 0 & 0 & \frac{1}{4} & -1 & 2 \\
\hline
\frac{13}{4}M - \frac{3}{2} & 0 & -M & \frac{3}{4}M - \frac{21}{2} & -M & 3M + 42
\end{array}$$

Notice that the second column is now cleared. We will repeat the methodology of the first two steps.

Example: Farmer's Problem

STEP 2: multiply row 3 by $\frac{4}{11}$, subtract $\frac{1}{4}$x row 3 from row 2, $\frac{1}{2}$x row 3 from row 1, $\frac{13}{4}M - \frac{3}{2}$ times row 3 from row 4

$$\begin{array}{ccccc|c}
0 & 0 & -1 & -\frac{6}{11} & \frac{2}{11} & \frac{7}{11} \\
0 & 1 & 0 & -\frac{5}{22} & \frac{1}{11} & \frac{9}{11} \\
1 & 0 & 0 & \frac{1}{11} & -\frac{4}{11} & \frac{8}{11} \\
\hline
0 & 0 & -M & \frac{5}{11}M - \frac{114}{11} & \frac{9}{4}M - \frac{6}{11} & \frac{7}{11}M + \frac{474}{11}
\end{array}$$

Now the first column is also cleared. We can keep going with this solution method. Since you know the ultimate answer, the remainder of the algorithm is left as an exercise.

Geometry of Simplex

Consider the manufacturer's problem.

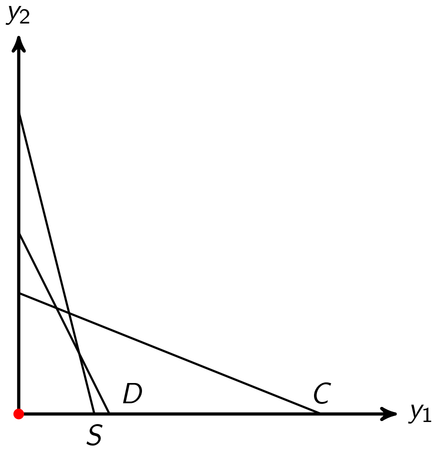
$$\max 24y_1 + 8y_2 \qquad s.t. \quad \begin{array}{lr} 2y_1 + 5y_2 \leq 40 & \text{Chopping} \\ 4y_1 + y_2 \leq 20 & \text{Slicing} \\ 10y_1 + 5y_2 \leq 60 & \text{Dicing} \\ y_1, y_2 \geq 0 & \end{array}$$

Geometry of Simplex

Set up the simplex array.

$$
\begin{array}{ccccc|c}
2 & 5 & 1 & 0 & 0 & 40 \\
4 & 1 & 0 & 1 & 0 & 20 \\
10 & 5 & 0 & 0 & 1 & 60 \\
\hline
-24 & -8 & 0 & 0 & 0 & 0
\end{array}
$$

Geometry of Simplex



We can graph this problem.

We know that our staring point is the origin.

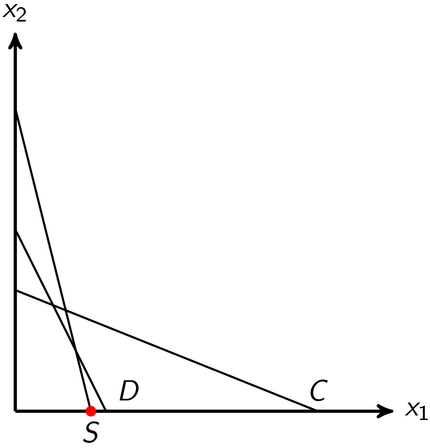
Geometry of Simplex

Now start by pivoting on the second row of the first column.

$$\begin{array}{cccccc|r}
0 & \frac{9}{2} & 1 & -\frac{1}{2} & 0 & & 30 \\
1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & & 5 \\
0 & \frac{5}{2} & 0 & -\frac{5}{2} & 1 & & 10 \\
\hline
0 & -2 & 0 & 6 & 0 & & 120
\end{array}$$

Geometry of Simplex

The simplex tableau now represents a solution of $x_1 = 5$, $\sigma_1 = 30$, and $\sigma_3 = 10$.



Geometry of Simplex

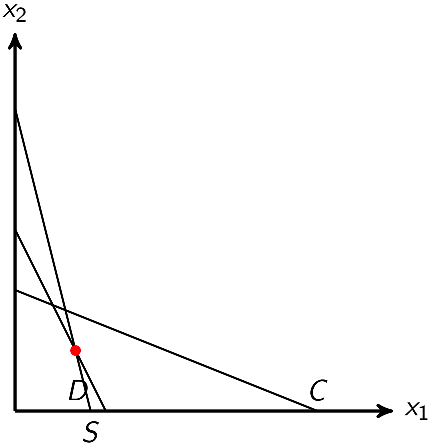
Proceeding by pivoting on the third row in the second column...

$$\begin{array}{ccccc|c}
0 & 0 & 1 & 4 & -\frac{9}{5} & 12 \\
1 & 0 & 0 & 0 & -\frac{1}{10} & 4 \\
0 & 1 & 0 & -1 & \frac{2}{5} & 4 \\
\hline
0 & 0 & 0 & 4 & \frac{4}{5} & 128
\end{array}$$

This obviously represents a solution.

Geometry of Simplex

The simplex tableau now represents a solution of $x_2 = 4$, $x_2 = 4$,



and $\sigma_1 = 12$.

Duality Theorems

We have two duality theorems for linear programs.

1. The optimal values of the primal and dual objective functions is always identical, provided that optimal feasible solutions exist. (Note that this is the case for the farmer's problem.)
2. If a certain *choice* variable in a linear program is optimally non-zero, then the corresponding *slack* variable in the dual program must be equal to zero. The converse also holds.

A Mechanical Explanation

A foolproof checklist to construct the dual of an LP problem.

1. Direction of optimization switches: i.e., max becomes min and vice versa.
2. Inequalities of constraints are reversed for geometric compliance of the constraint set, except non-negativity constraints.
3. Rows of constraint matrix in primal are transposed to columns of constraints in dual.
4. Row vector of coefficients in primal objective transposed to column vector of constants in dual constraints.
5. Column vector of constants in primal constraints transposed to row vector of coefficients in dual objective.
6. Primal decision variables become dual slack/surplus variables; primal slack/surplus variables become dual choice variables.

Side-by-side comparison

PRIMAL		DUAL	
max	$\pi = \mathbf{c}'\mathbf{x}$	min	$\pi^* = \mathbf{r}'\mathbf{y}$
s.t.	$\mathbf{Ax} \leq \mathbf{r}$	*s.t*	$\mathbf{A}'\mathbf{y} \geq \mathbf{c}$
	$\mathbf{x} \geq \mathbf{0}$		$\mathbf{y} \geq \mathbf{0}$
min	$\mathbf{C} = \mathbf{c}'\mathbf{x}$	max	$\mathbf{C}^* = \mathbf{r}'\mathbf{y}$
s.t.	$\mathbf{Ax} \geq \mathbf{r}$	*s.t*	$\mathbf{A}'\mathbf{y} \leq \mathbf{c}$
	$\mathbf{x} \geq \mathbf{0}$		$\mathbf{y} \geq \mathbf{0}$

Side-by-side comparison

Duality changes an n-dimensional choice subject to m constraints into an m-dimensional choice subject to n constraints.

$$\begin{array}{lllr}
\max & b_1x_1 + \cdots + b_nx_n & s.t. & a_{11}x_1 + \cdots + a_{1n}x_n \leq c_1 \\
 & & & a_{m1}x_1 + \cdots + a_{mn}x_n \leq c_m \\
 & & & x_1 \ldots x_n \geq 0
\end{array}$$

$$\begin{array}{lllr}
\min & c_1y_1 + \cdots + c_my_m & s.t. & a_{11}y_1 + \cdots + a_{1m}y_m \geq b_1 \\
 & & & a_{n1}y_1 + \cdots + a_{nm}y_m \geq b_n \\
 & & & y_1 \ldots y_m \geq 0
\end{array}$$

Shadow Values

We know that slack and surplus variables tell us how much unused capacity there is at the optimum with respect to each of our constraints. This is different from telling us the *value* of the constraints, as a Lagrangian multiplier would. However, we are able to glean comparable information from a linear program, particularly if we solve it using the simplex algorithm.
A slack or surplus variable that enters the basic solution will have a pivot, and therefore a value of zero in the bottom row. However, slack/surplus variables not in the basis have a value of zero, which means those constraints bind. In general, the objective function will be altered by changing the constraint, and there will be a non-zero shadow value for a binding constraint.

Shadow Values in Farmer's Problem

The farmer's problem yields the following final simplex tableau.

$$\begin{array}{ccccc|c}
0 & 0 & -\frac{11}{2} & \frac{5}{2} & 1 & \frac{7}{2} \\
0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\
1 & 0 & -2 & 1 & 0 & 2 \\
\hline
0 & 0 & -3 & -9 & 0 & 45
\end{array}$$

The harvesting constraint is slack, with a surplus variable value of $\frac{7}{2}$. The planting constraint binds, and the shadow value of the planting constraint is -3; expanding planting capacity by one unit would reduce costs by 3. The irrigating constraint binds. The shadow value of an additional unit of irrigating capacity is -9. A reasonable conclusion is that the farmer would get higher return on investment in expansion of irrigating as opposed to planting capacity, and no return at all on investment in harvesting capacity.

Shadow Values in Manufacturer's Problem

The manufacturer's problem had the following final simplex tableau.

$$\begin{array}{ccccc|c}
0 & 0 & 1 & 4 & -\frac{9}{5} & 12 \\
1 & 0 & 0 & 0 & -\frac{1}{10} & 4 \\
0 & 1 & 0 & -1 & \frac{2}{5} & 4 \\
\hline
0 & 0 & 0 & 4 & \frac{4}{5} & 128
\end{array}$$

We know that $y_1 = y_2 = 4$, and that the slack variable on the chopping constraint takes a value of 12. The slicing and dicing constraints bind. Clearly, relaxing the chopping constraint (by adding additional capacity) will not affect the maximum profit. However, relaxing the slicing constraint by one unit will increase profits by 4, and relaxing the dicing constraint will lead to an improvement in the objective function of $\frac{4}{5}$.

Using Shadow Values

The optimal value of the objective function (which is equal in the primal and dual) can be decomposed into a sum of shadow value times resource values. Given the following primal and dual problems

$$\max \quad f = b_1x_1 + \cdots + b_nx_n \qquad s.t. \quad \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n \leq c_1 \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq c_m \\ x_1 \ldots x_n \geq 0 \end{array}$$

$$\min \quad g = c_1y_1 + \cdots + c_my_m \qquad s.t. \quad \begin{array}{l} a_{11}y_1 + \cdots + a_{1m}y_m \geq b_1 \\ a_{n1}y_1 + \cdots + a_{nm}y_m \geq b_n \\ y_1 \ldots y_m \geq 0 \end{array}$$

We have the following relationship.

$$f^* = \sum_i^n b_i \cdot x_i^* = \sum_j^m c_j \cdot y_j^* = g^*$$

Farmer's Problem—Primal

We have written the primal form of the farmer's problem—cost minimization subject to production constraints.

$$\min 12W + 42B \qquad s.t. \quad \begin{aligned} W + 2B &\geq 3 \\ W + 4B &\geq 4 \\ 3W + B &\geq 3 \\ W, B &\geq 0 \end{aligned}$$

Farmer's Problem—Dual

We know the answer: $W^* = 2\,B^* = \frac{1}{2}\,C^* = 45$. We can also write the dual problem—production maximization subject to cost constraints for each crop.

$$\max 3P + 4I + 3H \qquad s.t. \quad \begin{aligned} &P + I + 3H \leq 12 \\ &2P + 4I + H \leq 42 \\ &P, I, H \geq 0 \end{aligned}$$

Solving this dual problem, we find that $P^* = 3,\ I^* = 9,\ H^* = 0,\ f^* = 45$. Notice that the value of the objective function is the same as in the primal case—$C^* = f^* = 45$. So the first duality theorem holds.

Farmer's Problem—Dual

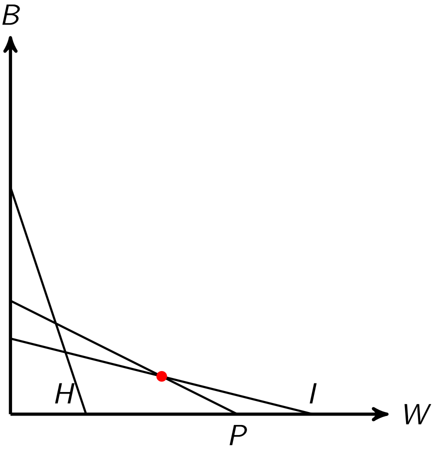
The second duality theorem also holds. Since the choice variables are different in the dual and primal (note difference with duality in nonlinear programs here), we pay attention to values of slack variables in the primal and choice variables in the dual. In the primal, harvesting was slack—in the dual the optimal choice of harvesting is zero. In the primal, both irrigating and planting constraints were binding. In the dual, the optimal chocies of each of the corresponding dual variables are non-zero.

Farmer's Problem—Comparison

PRIMAL	DUAL
$C^* = 45$	$f^* = 45$
$W^* = 2$	$\tau_1 = 0 \rightarrow W^* = 12$
$B^* = \frac{1}{2}$	$\tau_2 = 0 \rightarrow B^* = 42$
$\sigma_1 = 0 \rightarrow P^* = 3$	$P^* = 3$
$\sigma_2 = 0 \rightarrow I^* = 4$	$I^* = 9$
$\sigma_3 = \frac{7}{2} \rightarrow H^* = 6\frac{1}{2}$	$H^* = 0$

Interpreting these results, the cost-minimizing production vector is two units of W and one-half unit of B. In this case the planting and irrigating constraints bind but the harvesting constraint is slack, and the total cost is 45. The production-maximizing solution is to plant 3 units and irrigate 9. However, no harvesting is done—the production is maximized, not the profit. In this case the maximum production is also equal to 45, and the W constraint binds. The B constraint is slack.

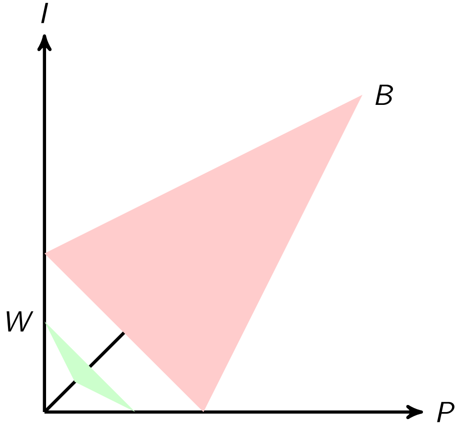
Geometric Duality



The Primal

Geometric Duality

The Dual



Shadow Values in Farmer's Dual

Note that the optimal choice variables in the dual problem ($P^* = 3$, $I^* = 9$, $H^* = 0$) correspond to the shadow values of the constraints in the primal? What are the shadow values of the constraints in the dual problem? We know that both constraints bind. As it turns out, the shadow value of the W constraint is 2, and the shadow value of the B constraint is $\frac{1}{2}$. These are exactly the optimal choices in the primal.

We have identified one further connection between the primal and the dual. Not only are the objective values identical, the complementary slackness of primal slack variables and dual choice variables (and vice versa). The primal shadow values are the dual choice variables, and vice versa.

Duality in Manufacturer's Problem

Recall our manufacturer's problem.

$$\max 24y_1 + 8y_2 \qquad s.t. \quad \begin{aligned} 2y_1 + 5y_2 &\leq 40 && \text{Chopping} \\ 4y_1 + y_2 &\leq 20 && \text{Slicing} \\ 10y_1 + 5y_2 &\leq 60 && \text{Dicing} \\ y_1, y_2 &\geq 0 \end{aligned}$$

We can now construct the dual—cost minimization subject to production constraints.

$$\min 40x_1 + 20x_2 + 60x_3 \qquad s.t. \quad \begin{aligned} 2x_1 + 4x_2 + 10x_3 &\geq 24 && \mathrm{y}_1 \\ 5x_1 + x_2 + 5x_3 &\geq 8 && \mathrm{y}_2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Duality in Manufacturer's Problem

The solution to the dual problem is $x_1 = 0$, $x_2 = 4$, $x_3 = \frac{4}{5}$. These values imply that both surplus variables are equal to zero, or that both constraints bind. Furthermore, $f^* = 128$.

The first duality theorem holds–the optimal value of the objective function is identical in the primal and the dual.

The second duality theorem clearly holds. The chopping constraint was slack in the primal, so the optimal dual choice is zero chopping; the slicing and dicing constraints had zero slack in the primal, so in the dual the corresponding choice variables have non-zero values.

Duality in Manufacturer's Problem

We can make a direct comparison on the primal and dual in this case.

PRIMAL	DUAL
$\Pi^* = 128$	$c^* = 128$
$y_1^* = 4$	$\tau_1 = 0 \rightarrow y_1^* = 24$
$y_2^* = 4$	$\tau_2 = 0 \rightarrow y_2^* = 8$
$\sigma_1 = 12 \rightarrow C^* = 28$	$x_1^* = C^* = 0$
$\sigma_2 = 0 \rightarrow S^* = 20$	$x_2^* = S^* = 4$
$\sigma_3 = 0 \rightarrow D^* = 60$	$x_3^* = D^* = \frac{4}{5}$

Shadow Values in Manufacturer's Dual

What are the shadow values of the manufacturer's dual problem? Solving the dual, we found that the dual choice variables were equal to the shadow values of the binding constraints in the primal. In the dual, the shadow values of the two binding constraints are each 4. It is not a coincidence that these shadow values are the same as the optimal choices in the primal problem.

Checking Shadow Values

Earlier we noted that optimal resource values times shadow prices returned the optimal objective function.

$$f^* = \sum_i^n b_i \cdot x_i^* = \sum_j^m c_j \cdot y_j^* = g^*$$

Check this relationship for the manufacturer's problem.

$$\Pi^* = 128 = 40 \cdot 0 + 20 \cdot 4 + 60 \cdot \frac{4}{5} = 24 \cdot 4 + 8 \cdot 4 = 128 = c^*$$

Check this relationship for the farmer's problem.

$$C^* = 45 = 3 \cdot 3 + 4 \cdot 9 + 3 \cdot 0 = 12 \cdot 2 + 42 \cdot \frac{1}{2} = 45 = f^*$$

Final Example

Consider a final example.

$$\max_{\mathbf{x}} f = 5x_1 + 3x_2 \qquad s.t. \quad \begin{aligned} 6x_1 + 2x_2 &\leq 36 \\ 5x_1 + 5x_2 &\leq 40 \\ 2x_1 + 4x_2 &\leq 28 \\ x_1, x_2 &\geq 0 \end{aligned}$$

We will solve the primal, then, the dual, then compare. The dual formulation is:

$$\min_{\mathbf{y}} g = 36y_1 + 40y_2 + 28y_3 \qquad s.t. \quad \begin{aligned} 6y_1 + 5y_2 + 5y_2 &\geq 5 \\ 2y_1 + 5y_2 + 4y_2 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Final Example—Primal

The final simplex tableau for the primal problem is:

$$
\begin{array}{ccccc|c}
1 & 0 & \frac{1}{4} & -\frac{1}{10} & 0 & 5 \\
0 & 1 & -\frac{1}{4} & \frac{3}{10} & 0 & 3 \\
0 & 0 & \frac{1}{2} & -1 & 1 & 6 \\
\hline
0 & 0 & \frac{1}{2} & \frac{2}{5} & 0 & 34
\end{array}
$$

We can read off our solution: $x_1^* = 5$, $x_2^* = 3$, $\sigma_1^* = \sigma_2^* = 0$, $\sigma_3^* = 6$, and $f^* = 34$. The shadow value of the first constraint is $\frac{1}{2}$; the shadow value of the second constraint is $\frac{2}{5}$.

Final Example—Dual

The final simplex tableau for the dual problem is:

$$\begin{array}{ccccc|c}
1 & 0 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
0 & 1 & 1 & \frac{1}{10} & -\frac{3}{10} & \frac{2}{5} \\
\hline
0 & 0 & -6 & -5 & -3 & 34
\end{array}$$

We can read off our solution: $y_1^* = \frac{1}{2}$, $y_2^* = \frac{2}{5}$, $y_3^* = 0$, $\tau_1^* = \tau_2^* = 0$, and $g^* = 34$. The shadow value of the first constraint is -6; the shadow value of the second constraint is -5; the shadow value of the third constraint is -3.

Final Example—Comparison

Now make a direct comparison of the primal and dual.

PRIMAL		**DUAL**	
$f^* = 128$		$g^* = 128$	
$x_1^* = 5$		$\tau_1 = 0$	shadow value=-5
$x_2^* = 3$		$\tau_2 = 0$	shadow value=-3
$\sigma_1 = 0$	shadow value $=\frac{1}{2}$	$y_1^* = \frac{1}{2}$	
$\sigma_2 = 0$	shadow value $=\frac{2}{5}$	$y_2^* = \frac{2}{5}$	
$\sigma_3 = 6$	shadow value $=0$	$y_3^* = 0$	(shadow value=-6)

This confirms our earlier results specific to linear programs, and reminds us that one useful feature of duality relationships in both linear and nonlinear settings is that we can solve a given problem in either way, whichever proves to be more expedient.

Matching Pennies

Consider a game with two players earning payoffs as a result of independent flips of a penny. Each player decides on a strategy and reveals his penny to the other, then payoffs are calculated. The payoff matrix for the row player is:

$$\begin{array}{lcc} & \textit{Heads} & \textit{Tails} \\ \textit{Heads} & -10 & 20 \\ \textit{Tails} & 10 & -10 \end{array}$$

where the column heading index the outcome of the column player's play. Indexing pairs as (row payoff, column payoff), we can express the whole game as:

$-10, 10$	$20, -20$
$10, -10$	$-10, 10$