Appendix 2

Proof of Proposition 1

Using the relationships (4) and (5), and the assumption that ability is distributed uniformly on the interval \([0, A]\) we obtain

\[
L_t^{MP} = \int_{0 \leq a(\omega) \leq a_t^{*MP}} l_t^{MP}(\omega)d\mu_t(\omega) = CE_{t-1}^{MP} \int_0^A \frac{a}{A} da = \frac{(a_t^{*MP})^2}{2A} CE_{t-1}^{MP}
\]

(A1)

\[
H_t^{MP} = \int_{a_t^{*MP} \leq a(\omega) \leq A} h_t^{MP}(\omega)d\mu_t(\omega) = \int_{a_t^{*MP}}^A \left[(b + B)CE_{t-1}^{MP} a - Bh^*\right] \frac{1}{A} da = \\
= \frac{(b + B)CE_{t-1}^{MP}}{2A} \left[A^2 - (a_t^{*MP})^2\right] - \frac{Bh^*}{A} \left[A - a_t^{*MP}\right]
\]

(A2)

Now substituting these expressions in the tax revenue formula (10) and using the expression (6) for \(a_t^{*MP}\) we can transform the equation (11) into the following:

\[
q_t^{MP} = \frac{\tau_t (b + B) ACE_{t-1}^{MP}}{2} J_t^{MP}
\]

(A3)

where

\[
J_t^{MP} = \left[1 - \frac{2Bh^*}{(b + B) ACE_{t-1}^{MP}} + \frac{\theta_t (Bh^*)^2}{\left(\theta_t (b + B) - 1\right)(b + B)(ACE_{t-1}^{MP})^2}\right]
\]

(A4)

Similar to the derivation of the expression (A2), we can rewrite the expression (14) as

\[
q_t^{MP} = \int_{a_t^{*MP}} \left[(b + B)CE_{t-1}^{MP} a - Bh^*\right] A^{-1} da = \frac{(b + B)CE_{t-1}^{MP}}{2A} \left[\left(a_t^{MP}\right)^2 - (a_t^{*MP})^2\right] - \frac{Bh^*}{A} \left[a_t^{MP} - a_t^{*MP}\right]
\]

Using the relationship (15) as well as (6), this can be rewritten as

\[
2q_t^{MP} = (z_t^{MP})^2 (b + B) ACE_{t-1}^{MP} + z_t^{MP} \frac{2Bh^*}{\theta_t (b + B) - 1}
\]

(A5)

or equivalently, according to (A3), as

\[
(z_t^{MP})^2 (b + B) ACE_{t-1}^{MP} + z_t^{MP} \frac{2Bh^*}{\theta_t (b + B) - 1} - \tau_t (b + B) ACE_{t-1}^{MP} J_t^{MP} = 0
\]

(A6)

It is clear that the quadratic equation (A6), in terms of \(z_t^{MP}\), has a unique non-negative solution. Therefore according to the expression (A4) it uniquely defines \(z_t^{MP}\) as a function of the
prior period’s education quality $E_i^{MP}$. The quantity of teachers $z_t^{MP}$ along with their quality $q_t^{MP}$ expressed by (A5) thus determine period $t$ quality of basic education $E_t^{MP}$ according to formula (8). This completes an iteration of the recursion which uniquely defines the recursive dynamic equilibrium (RDE) in our model under the merit pay regime for teachers. □

**Basic Education Quality Optimization Problem under the Collective Bargaining Regime**

Thanks to Assumption 1 of the uniform distribution of innate ability on the interval $[a, A]$ and according to the basic and advanced education production functions (4) and (5) we can simplify expressions (8) and (17), respectively, as

$$E_i^{CB} = (z_i^{CB})^\nu (q_i^{CB})^\nu = \frac{z_i^\nu \left[ \bar{h}_i^{2} - h_i^{2} \right]^{\nu}}{[2(b + B)ACE_{i-1}^{CB}]}$$

$$z_t^{CB} = \frac{\bar{h}_t^{CB} - h_t^{CB}}{(b + B)ACE_{i-1}^{CB}}$$

(A7)

and therefore problem (20) to maximize the quality of basic education $E_t$ subject to the budget constraint can be restated as

$$\max_{z, h} \frac{z_i^\nu \left[ \bar{h}_i^{2} - h_i^{2} \right]^{\nu}}{[2(b + B)ACE_{i-1}^{CB}]}^{\nu}$$

subject to (A1)

$$z_i^{CB} \theta_i \bar{h}_i^{CB} = T_i^{CB}$$

$$a_i^{CB} \geq a_i^{*CB}$$

or equivalently, according to (A7), as

$$\max_{z, h} 2^{\nu} \left( \bar{h}_i + h_i \right)^{\nu} z_i^{\nu+\nu}$$

subject to $z_i^{CB} \theta_i \bar{h}_i^{CB} = T_i^{CB}$

$$a_i^{CB} \geq a_i^{*CB}$$

(A8)

Note that the optimal lower and upper cut-off levels of teachers’ human capital $\bar{h}_i^{CB}$, $h_i^{CB}$ are related through the optimal choice of their number $z_i^{CB}$ according to equation (A7). The
optimization in problem (A8) thus expresses the trade-off between the quantity and quality of teachers to be hired. The quality of the top teacher $\tilde{h}_t$ will not only determine his salary $I^h_t = \theta_t w_t \tilde{h}_t$ due to his outside option as a skilled worker, but will set the identical salary for all other teachers according to the equal pay based collective bargaining agreement. Conversely, teacher salary $I^h_t$ set by the education agency will uniquely determine the top teacher quality $\tilde{h}_t$. Therefore the total teachers’ wage bill in the education budget constraint is given by $z_t \theta_t \bar{w} \tilde{h}_t$.

Therefore using relationships (18) to express $\tilde{h}_t$ and $h_t$ and then eliminate $a_t$ according to formula (19), we obtain

\[ q^{CB}_{t} = \frac{\tilde{h}_t^2 - h_t^2}{2(b + B)ACE_{t-1}^{CB}} = \left[ (b + B)CE_{t-1}^{CB} \tilde{a}_t^{CB} - Bh^* - \frac{1}{2} z_t^{CB} (b + B) ACE_{t-1}^{CB} \right] z_t^{CB} \]  

so we can restate the education quality optimization problem (A9) as

\[
\max_{z_t, \tilde{h}_t} E_t^{CB} = 2^{-v} \left[ 2(b + B)CE_{t-1}^{CB} \tilde{a}_t^{CB} - 2 Bh^* - z_t^{CB} (b + B) ACE_{t-1}^{CB} \right] z_t^{CB} 
\]

subject to $z_t^{CB} \theta_t w_t \left[ (b + B)CE_{t-1}^{CB} \tilde{a}_t^{CB} - Bh^* \right] = T_t^{CB}$ and

\[ \tilde{a}_t^{CB} - A z_t^{CB} \geq a_t^* \]  

(A10)

**Assumptions of the Model**

We will now spell out specific conditions on the model’s parameters behind the Assumption 2 outlined in Section 4 of the paper. We impose the following restrictions on the economy’s parameters, where $E_{-1}$ is an exogenously given per student basic education quality provided to generation $G_0$ individuals.

**Assumption 2.** The returns to the human capital of teachers are non-decreasing: $v \geq 1$.

Furthermore, the following inequalities are true for $t = 0, 1, \ldots$:

\[(i) \quad \left( \frac{\nu}{\gamma} (b + B) AC (1 - \tau_t) \right)^{1/2} \left( \frac{\gamma}{2v + \gamma} \right)^{1/2} \left( \frac{\tau_t}{1 - \tau_t} \right)^{1/2} \left( 1 - \frac{B h^*}{(b + B) AC E_{-1}} \right) > 1 \]

\[(ii) \quad \left( \frac{\nu (1 - \tau_t)}{\gamma} - \frac{1}{2} \right) \left( \frac{\gamma}{2v + \gamma} \right)^{1/2} \left( \frac{\tau_t}{1 - \tau_t} \right)^{1/2} \left( 1 - \frac{B h^*}{(b + B) AC E_{-1}} \right) > \frac{1}{\bar{\theta}_t (b + B)} \]
The main thrust of the above conditions concerns the parameters which characterize educational gains. Inequality (i) is satisfied, if parameter $C$ characterizing the human capital gains in basic education according to (4) is sufficiently large. Inequality (ii) will hold if $(b + B)$, a productivity characteristic of the college education production function (5), is large enough.

Assumption 2 also requires that education taxes $\tau_i$ were not too small (the above inequalities imply a uniform lower bound for $\tau_i$) while not exceeding $1 - \frac{\gamma}{2\nu}$. Indeed, $\tau_i \leq 1 - \frac{\gamma}{2\nu}$ must be true in order for the term $\left(\frac{\nu(1 - \tau_i)}{\gamma} - \frac{1}{2}\right)$ in the above condition (ii) to be positive. As discussed in Section 4, the inequality $\tau_i \leq 1 - \frac{\gamma}{2\nu}$ imposes a requirement that $\gamma$, the relative importance of the teacher-student ratio for schooling effectiveness, should not be substantially greater than $\nu$, the relative importance of the teacher quality.

We will now proceed to proving Proposition 2, the Lemmas, Corollaries, and Theorems. We will first prove Proposition 2 and Lemmas 1 and 2 under the hypothesis that Lemma 3 is correct, i.e. that the cut-off ability $a^*_{CB}$ of college attendees satisfies formula (6). We will then prove that Lemma 3 is indeed correct in the recursive dynamic equilibrium, and thereby the imposition of the hypothesis does not diminish the generality of (or create circularity problems with) the argument.

**Proof of Proposition 2**

According to the teacher salary equation (16) and the tax revenue formula (21), the budget constraint in the education quality optimization problem can be rewritten as

$$(1 - \tau_i)z^{CB}_i \theta^C\bar{h}^{CB}_i = \tau_i \left(L_i + \theta^C H^\tau_i\right) \quad (A11)$$

Using the education production functions (4) and (5) and the assumption that innate ability is uniformly distributed on $[0, A]$ we can express the aggregate supply of unskilled and skilled labor, respectively, as

$$L_i = \int_{0 \leq a(\omega) \leq a^*_i} l_i(\omega) d\mu_i(\omega) = CE_{t-1}^{CB} \int_0^{a^*_i} \frac{a}{2A} da = \frac{(a^*_i)^2}{2A} CE_{t-1}^{CB} \quad (A12)$$
Therefore expressing $\bar{h}_i$ through $\bar{a}_i$ according to the relationship in (18) we can rewrite the budget constraint (A5) as

\[
(1-\tau_i) \theta z_i CB \left( (b + B) CE_{i-1} \bar{a}_{i CB} - Bh^* \right) = 
\]

\[
\frac{\tau_i CE_{i-1}}{2A} \left( (a_i^{* CB})^2 + \theta_i (b + B) \left( A^2 - (a_i^{* CB})^2 - (\bar{a}_i^{CB})^2 \right) \right) - \frac{\tau_i \theta_i Bh^*}{A} \left[ A - a_i^{* CB} - \bar{a}_i^{CB} + a_i^T CB \right] 
\]

(A14)

We now eliminate variables $a_i^*$ and $a_i$ from (A8) by substituting the value of $a_i^*$ given by (6) according to Lemma 3, and using the expression (19). This immediately turns expression (A14) into a linear equation in terms of variable $\bar{a}_i$, which yields

\[
\bar{a}_i^{CB} = \frac{z_i^{CB} \tau_i A}{2} + \frac{Bh^*}{(b + B) CE_{i-1}} + \frac{\tau_i A J_i^{CB}}{2z_i^{CB}} 
\]

(A15)

where $J_i^{CB}$ is as defined by formula (23). Expression (A15) incorporates the education budget constraint of the optimization problem (A10). That problem’s objective function, upon substituting the expression (A11) for $\bar{a}_i$, becomes a function of a single variable $z_i$. We will first solve for its unconstrained maximization and then discuss the verification that its solution satisfies the only remaining constraint $\bar{a}_i^{CB} - A z_i^{CB} \geq a_i^{* CB}$ in the optimization problem (A10).

Thus we are looking at the unconstrained maximization of the following function:

\[
F(z_i^{CB}) = q_i^* z_i^\gamma = \left( \frac{\tau_i (b + B) AC E_{i-1}^{CB}}{2} - \tau_i Bh^* + \frac{\tau_i \theta_i B h^2}{2(\theta_i (b + B) - 1) AC E_{i-1}^{CB}} \right) \left( 1 - \tau_i \right) (b + B) AC E_{i-1}^{CB} z_i^{z_i} 
\]

Its first order necessary condition is given by the equation

\[
\gamma z_i^{\gamma-1} q_i^* - \nu (1 - \tau_i) (b + B) AC E_{i-1} z_i^{\gamma+1} q_i^{\gamma-1} = 0 
\]

(A16)

yielding unique non-negative solution:

\[
z_i^{CB} = \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_i}{1 - \tau_i} \right) \left( J_i^{CB} \right)^{1/2} 
\]

(A17)

It is straightforward to verify that this solution also satisfies the second order sufficient condition.
of the maximization problem.

Substituting (A17) in (A16) we obtain the expression for the aggregate teacher quality:

\[ q_t^{CB} = \frac{\nu \tau_t (b + B) ACE_{t-1}^{CB}}{2 \nu + \gamma} J_t^{CB} \]  

(A18)

Substituting expression (A17) back into formula (A15) we obtain

\[ \bar{a}_t^{CB} = \frac{\tau_t A z_t^{CB} + B h^*}{2} + \frac{B h^*}{(b + B) CE_{t-1}^{CB}} + \frac{\tau_t A}{2 z_t^{CB}} J_t^{CB} \]

which simplifies, by using equation (A17) again, into

\[ \bar{a}_t^{CB} = \frac{B h^*}{(b + B) CE_{t-1}^{CB}} + A z_t^{CB} \left( \frac{\nu (1 - \tau_t)}{\gamma} + \frac{1}{2} \right) \]  

(A19)

Combining this with (19) we obtain

\[ \bar{a}_t^{CB} = \frac{B h^*}{(b + B) CE_{t-1}^{CB}} + A z_t^{CB} \left( \frac{\nu (1 - \tau_t)}{\gamma} - \frac{1}{2} \right) \]  

(A20)

As discussed earlier, in order to ascertain that the expressions (A17), (A19), (A20) represent the solution of the constrained optimization problem (A10), it remains to verify that the constraint \( q_t > a_t^* \) does hold for \( a_t^* \) given by (A20). This will be indeed demonstrated in the proof of Lemma 2 below.

Observe that the education policy optimization as well as the individuals’ and the production sector’s general equilibrium reactions are determined recursively. Indeed, according to expressions (A17), (A19), and (A20), education quality \( E_{t-1}^{CB} \) uniquely determines optimal education policy in period \( t \), i.e. the number of teachers, as well as the range of their innate abilities and thereby, due to (18), the range of their human capital attainment. This in turn will uniquely determine college attendance and employment decisions by generation \( t \) individuals, hence their incomes and their allocations. The education agency’s policy will also determine the current period’s basic education quality \( E_t \), so the recursion continues. □

**Proof of Lemma 1**

According to (A4), \( E_t^{CB} = \left( z_t^{CB} (b + B) CE_{t-1}^{CB} \bar{a}_t^{CB} - z_t^{CB} B h^* - \frac{1}{2} (b + B) ACE_{t-1}^{CB} (z_t^{CB})^2 \right)^v (z_t^{CB})^\gamma \).

Substituting the expression for \( \bar{a}_t \) in (A19), we obtain
\[ E_{t}^{\text{CB}} = \left( A(b + B)CE_{t-1}^{\text{CB}} \left( \frac{v(1-\tau_{t})}{y} \right) \right)^{v} \left( z_{t}^{\text{CB}} \right)^{2v+y}, \text{ or according to } (A11) \]

\[ E_{t}^{\text{CB}} = \left( A(b + B)CE_{t-1}^{\text{CB}} \left( v(1-\tau_{t}) \right) \right)^{v} \left( \frac{\gamma}{2v+y} \frac{\tau_{t}}{1-\tau_{t}} J_{t}^{\text{CB}} \right)^{v+y/2} \]

Note that since \( \frac{\theta_{i}(b + B)}{\theta_{i}(b + B) - 1} > 1 \), the following inequality is true

\[ J_{t}^{\text{CB}} > \left[ 1 - \frac{Bh^{*}}{(b + B) ACE_{t-1}^{\text{CB}}} \right]^{2} \]

(A21)

Therefore we can write

\[ E_{t}^{\text{CB}} > \left( \frac{V}{\gamma} (b + B) ACE_{t-1}^{\text{CB}} (1-\tau_{t}) \right)^{v} \left( \frac{\gamma}{2v+y} \frac{\tau_{t}}{1-\tau_{t}} \right)^{v+y/2} \left( 1 - \frac{Bh^{*}}{(b + B) ACE_{t-1}^{\text{CB}}} \right)^{2v+y} \]

Thus, in order to prove the Lemma it is sufficient to show that for all \( t = 0,1,... \)

\[ \left( \frac{V}{\gamma} (b + B) AC (1-\tau_{t}) \right)^{2v+y/v} \left( \frac{\gamma}{2v+y} \frac{\tau_{t}}{1-\tau_{t}} \right)^{2v+y/2} \left( 1 - \frac{Bh^{*}}{(b + B) ACE_{t-1}^{\text{CB}}} \right) > 1 \]

which is indeed true according to Assumption 2(i) and by the induction argument. \( \square \)

**Proof of Lemma 2**

Based on Lemma 3 we use expression (6) for \( \alpha_{i}^{*}\text{CB} \). Then according to (A20) our task of proving the inequality \( \alpha_{i}^{*}\text{CB} > \alpha_{i}^{*}\text{CB} \) is equivalent to verifying the inequality

\[ \frac{Bh^{*}}{(b + B) CE_{t-1}^{\text{CB}}} + Az_{t}^{\text{CB}} \left( \frac{v(1-\tau_{t})}{\gamma} - \frac{1}{2} \right) > 1 \]

or

\[ Az_{t}^{\text{CB}} \left( \frac{v(1-\tau_{t})}{\gamma} - \frac{1}{2} \right) > \frac{1}{(b + B) CE_{t-1}^{\text{CB}}} \frac{Bh^{*}}{\theta_{i}(b + B) - 1} \]

Upon substituting the expression (A17) for \( z_{t} \), the last inequality becomes

\[ (A^{1/2})^{v+y/2} \left( \frac{\gamma}{2v+y} \frac{\tau_{t}}{1-\tau_{t}} \right) \left( J_{t}^{\text{CB}} \right)^{v/2} > \frac{1}{(b + B) CE_{t-1}^{\text{CB}}} \frac{Bh^{*}}{\theta_{i}(b + B) - 1} \]

(A22)

Under Lemma 3 the right hand side in (A22) is smaller than \( \frac{A}{\theta_{i}(b + B)} \) since \( \alpha_{i}^{*} < A \). Therefore
according to (A21) in order to prove the inequality (A22) it is by far sufficient to establish

\[
\left( \frac{\nu(1-\tau_c)}{\nu} - \frac{1}{2} \left( \frac{1-\tau_c}{2\nu + \gamma 1-\tau_c} \right)^{1/2} \left( 1 - \frac{Bh^*}{(b+B)\Lambda(CE_{t-1}^{CB})} \right) \right) > \frac{1}{\theta_i(b+B)}
\]

(A23)

which is indeed true for all \( t = 0,1,\ldots \) according to Assumption 2(ii) combined with Lemma 1. \( \Box \)

**Proof of Lemma 3**

The above proofs were based on the hypothesis that Lemma 3 is correct, i.e., that the ability cut-off for college attendance \( a_i^* \) satisfies equality (6). Thus we have proved that if the college attendance cut-off ability is \( a_i^{CB} = \frac{1}{CE_{t-1}^{CB}} \frac{\theta_i Bh^*}{\theta_i(b+B)-1} \) then the optimal education policy requires that all teachers’ ability strictly exceed this threshold. This in turn means that the marginal college graduate will be employed in the production sector. As we explained after stating the equality (6), if an individual with ability below \( a_i^* \) attended college, his skilled human capital adjusted for the net productivity augmentation \( \theta_i \) would be inferior to his unskilled human capital derived from the first stage of education, therefore a job in production sector’s skilled labor force would not compel such individual to attend college. Thus the only way the violation of Lemma 3 could occur is if such individual had an opportunity to be hired as a teacher.

Compare, however, the optimization problem (20) where \( a_i^{CB} < \frac{1}{CE_{t-1}^{CB}} \frac{\theta_i Bh^*}{\theta_i(b+B)-1} \) to the one with \( a_i^{*CB} = \frac{1}{CE_{t-1}^{CB}} \frac{\theta_i Bh^*}{\theta_i(b+B)-1} \). One can easily see that the only difference would be lower tax revenue \( T_{t}^{CB} \) in the former case. Therefore such education policy would be inferior to the one where \( a_i^{*CB} = \frac{1}{CE_{t-1}^{CB}} \frac{\theta_i Bh^*}{\theta_i(b+B)-1} \). Thus the latter indeed characterizes the recursive dynamic equilibrium optimum, i.e., Lemma 3 is correct. \( \Box \)

**Proof of Proposition 3**

Based on the income formulas (2)-(3), the human capital accumulation formulas (4)-(5) and using the uniform distribution of abilities as well as the formula (9) for the threshold ability between the groups, we can obtain the mean income of unskilled individuals:
and the mean income of the skilled (ignoring the distortion due to collective bargaining in the education sector):

\[ T^s_i = \frac{I^s_i(a^*_i) + I^s_i(A)}{2} = \frac{w_\theta Bh^*}{2\left(\theta_i(b + B) - 1\right)} = \frac{wCE_{t-1}^{CB}a^*_i}{2} \]

Thus the inequality between the groups can be characterized by

\[ \sigma^{s/u}_{i} = \frac{T^s_i}{T^u_i} = \frac{A(b + B)CE_{t-1}^{CB}(\theta_i(b + B) - 1)}{Bh^*(2 - \theta_i(b + B))} \]

This expression obviously increases in basic education quality, which according to Lemma 1 rises over time.

The inequality within the skilled group (ignoring the aforementioned distortion) is characterized by

\[ \sigma^s_i = \frac{I^s_i(A) - Bh^*}{I^s_i(a^*_i) - CE_{t-1}^{CB} - Bh^*} = \left(\theta_i(b + B) - 1\right)\frac{(b + B)ACE_{t-1}^{CB} - Bh^*}{Bh^*} \]

Using formula (6) we can rewrite this as

\[ \sigma^s_i = \frac{\theta_i(b + B)}{a^*_i}\left(1 - \frac{Bh^*}{CE_{t-1}^{CB}}\right) \]

which too grows with the rise of basic education quality. □

**Proof of Lemma 4**

By differentiating expressions (6) and (A17) we immediately obtain:

\[ \frac{\partial a^{*CB}_{i}}{\partial E_{t-1}^{CB}} = \frac{-\theta_i Bh^*}{\left(\theta_i(b + B) - 1\right)CE_{t-1}^{CB}^2} < 0 \quad (A24) \]

\[ \frac{\partial z^{CB}_{i}}{\partial E_{t-1}^{CB}} = \frac{1}{2\nu + \gamma}\left(\frac{\tau_i}{1 - \tau_i}\right)\frac{Bh^*}{(b + B)AC(E_{t-1}^{CB})^2} \left(1 - a^{*CB}_{i}/A\right) > 0 \quad (A25) \]

According to (A19) and (A20), respectively, we can write
\[
\frac{\partial \tilde{a}_{CB}}{\partial E_{t-1}^{CB}} = -\frac{B_{h}^{*}}{(b + B) C(E_{t-1}^{CB})^{2}} + A_{i}^{CB} \left( \frac{2\nu(1 - \tau_{i}) + \gamma}{2\nu + \gamma} \right) \left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right) \frac{B_{h}^{*}}{(b + B) A C(E_{t-1}^{CB})^{2}} \left( 1 - \frac{a_{*}^{CB}}{A} \right) \]
(A26)

\[
\frac{\partial a_{CB}}{\partial E_{t-1}} = -\frac{B_{h}^{*}}{(b + B) C(E_{t-1}^{CB})^{2}} + A_{i}^{CB} \left( \frac{2\nu(1 - \tau_{i}) - \gamma}{2\nu + \gamma} \right) \left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right) \frac{B_{h}^{*}}{(b + B) A C(E_{t-1}^{CB})^{2}} \left( 1 - \frac{a_{*}^{CB}}{A} \right) \]
(A27)

According to (A17) and (A21) we can write

\[
z_{i}^{CB} > \left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right)^{1/2} \left( 1 - \frac{B_{h}^{*}}{(b + B) A C(E_{t-1}^{CB})} \right) > \left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right)^{1/2} \left( 1 - \frac{\theta_{B} B_{h}^{*}}{(b + B) A C(E_{t-1}^{CB})} \right) \]

\[
= \left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right) \left( 1 - \frac{a_{*}^{CB}}{A} \right)
\]

Thus the expression (A26) will be negative as long as the inequality

\[
\left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right)^{1/2} > \left( \frac{\nu}{2\nu + \gamma} \right) \left( \frac{\tau_{i}}{1 - \tau_{i}} \right) \left( \frac{2\nu(1 - \tau_{i}) + \gamma}{2\nu + \gamma} \right)
\]
(A28)

is true, or equivalently

\[
\frac{\nu}{2\nu + \gamma} \frac{\tau_{i}}{1 - \tau_{i}} \left( \frac{\nu}{2\nu + \gamma} \frac{\tau_{i}^{2}}{1 - \tau_{i}} \right) \left( \frac{2\nu(1 - \tau_{i}) + \gamma}{2\nu + \gamma} \right)^{2}
\]
which can be simplified as:

\[
(1 - \tau_{i}) \left( \frac{2\nu}{\gamma} + 1 \right) > \tau_{i} \left( \frac{\nu}{\gamma} (1 - \tau_{i}) + \frac{1}{2} \right)^{2}
\]

This last inequality is certainly true if

\[
(1 - \tau_{i}) \left( \frac{2\nu}{\gamma} + 1 \right) > \tau_{i} \left( \frac{\nu}{\gamma} (1 - \tau_{i}) + \frac{1}{2} \right)^{2}
\]
which reduces to \( \tau_{i} < \frac{4}{5 + 2\nu / \gamma} \). The latter is guaranteed by the condition \( \tau_{i} < 1 - \frac{\nu}{2\nu} \), which is imposed by Assumption 2(ii). Thus we have proven the negativity of the expression (A26).

Comparing (A26) and (A27) one can see that negativity of (A26) implies the same for (A27). Therefore we can conclude that \( \frac{\partial \tilde{a}_{CB}}{\partial E_{t-1}^{CB}} < 0 \), \( \frac{\partial a_{CB}}{\partial E_{t-1}} < 0 \), completing the Lemma’s proof. \( \square \)
Proof of Corollary to Theorem 1

Recall that according to relationships (18)
\[ h^C_B = a^C_B (b + B) C E^C_{i-1} - B h^* \quad \text{and} \quad \overline{h}^C_B = a^C_B (b + B) C E^C_{i-1} - B h^* \]

Therefore due to (A19) and (A20), respectively, as well as to (A17) we can write
\[ \overline{h}^C_B = (b + B) A C E^C_{i-1} \left( \frac{v(1-\tau_i)}{\gamma} - \frac{1}{2} \right) z^C_i \]
\[ = (b + B) C A \left( \frac{v(1-\tau_i)}{\gamma} - \frac{1}{2} \right) \left( E^C_{i-1} \right)^{1/2} \left( \frac{2 B h^* E^C_{i-1}}{(b + B) A C} + \frac{\theta_i (B h^*)^2}{(\theta_i (b + B) - 1)(b + B)(A C)^2} \right) \]

\[ k^C_B = (b + B) A C E^C_{i-1} \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right) z_i \]
\[ = (b + B) A C \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right) \left( E^C_{i-1} \right)^{1/2} \left( \frac{2 B h^* E^C_{i-1}}{(b + B) A C} + \frac{\theta_i (B h^*)^2}{(\theta_i (b + B) - 1)(b + B)(A C)^2} \right) \]

both increasing functions of \( E^C_{i-1} \). Indeed, the derivative of each is proportionate to the expression \( \frac{\theta_i t_i B h^*}{(\theta_i (b + B) - 1) A C E^C_{i-1}} \), which is positive as shown by (A23). \( \square \)

Proof of Theorem 2

The proof will proceed by the induction argument.

Consider first the effect of a positive shock to coefficient \( \theta_i \) in period \( t = t_0 \) on education policy variables in this same period. According to (A17) direct derivative \( \frac{\partial E^C_{i-1}}{\partial \theta_i} \) has the same sign as \( \frac{\theta_i}{\theta_i (b + B) - 1} \), i.e. negative, so we can write
\[ \frac{\partial E^C_{i-1}}{\partial \theta_i} < 0 \] (A29)

Therefore, according to (A19) and (A20), respectively, we can write
\[ \frac{\partial a^C_B}{\partial \theta_i} = \frac{\partial a^C_B}{\partial z^C_i} \frac{\partial z^C_i}{\partial \theta_i} = A \left( \frac{3}{2} - \tau_i \right) \frac{\partial z^C_i}{\partial \theta_i} < 0 \] (A30)
Recall that according to the derivation of (A10)

\[ q_{t}^{CB} = \left( (b + B)CE_{t-1}^{CB} - Bh^{*} \right) - \frac{1}{2} z_{t}^{CB} A(b + B)CE_{t-1}^{CB} \]

Using formula (A19) we can rewrite the above as

\[ q_{t}^{CB} = \frac{v(1 - \tau_{t}) (b + B)ACE_{t-1}^{CB} (z_{t}^{CB})^{2}}{\gamma} \]  

(A32)

which implies, according to (A29), that

\[ \frac{\partial q_{t}^{CB}}{\partial \theta_{t}} < 0 \]  

(A33)

for \( t = t_{0} \). Combining (A29) with (A32) and referring to (8) we can conclude that for \( t = t_{0} \)

\[ \frac{\partial E_{t}^{CB}}{\partial \theta_{t}} < 0 \]  

(A34)

We can now proceed to the next step of the induction and evaluate the effect born by the education policy variables in period \( t = t_{0} + 1 \), keeping in mind two sources of this effect: the direct effect of higher value of \( \theta_{t+1} \) and the indirect one caused by lower education quality in the previous period \( E_{t} \) as established in (A34). The results in (A29), (A33) and (A34) show that the direct effects on the variables \( z_{t+1}, q_{t+1}, E_{t+1} \) in any period \( t + 1 \) of a contemporaneous rise in \( \theta_{t+1} \) are negative. For the purposes of completing the induction argument it will therefore be sufficient to prove that a decline in \( E_{t} \) will have a negative effect on \( z_{t+1}, q_{t+1}, E_{t+1} \), in other words that the derivatives of these variables with respect to \( E_{t} \) are all positive.

According to (A17) the derivative \( \frac{\partial (z_{t+1}^{2})}{\partial E_{t}} \) has the same sign as the expression

\[ 1 - \frac{\theta_{t+1} Bh^{*}}{(\theta_{t+1} (b + B) - 1) ACE_{t-1}^{CB}} \]

which according to Lemma 3 is equal to \( 1 - A^{-1} d_{t}^{*} \) and therefore is positive. Thus we can conclude that

\[ \frac{\partial z_{t+1}^{CB}}{\partial E_{t}^{CB}} > 0 \]  

(A35)
Rewriting expression (A12) for period \( t+1 \): \( q_{t+1}^{CB} = A(b + B) C E_t^{CB} \frac{\nu(1-\tau)}{\gamma} (z_{t+1}^{CB})^2 \), we note that a rise in \( E_t^{CB} \) affects \( q_{t+1}^{CB} \) directly (obviously positively) as well as indirectly through \( z_{t+1}^{CB} \), also positively according to (A35). We can therefore conclude that \( \frac{\partial q_{t+1}^{CB}}{\partial E_t^{CB}} > 0 \). This combined with (A35) implies due to (8) that \( \frac{\partial E_{t+1}^{CB}}{\partial E_t^{CB}} > 0 \). Thus according to the above discussion the Theorem’s proof is complete. □

**Proof of Theorem 3**

The proof will proceed by induction.

**Segment (a).** Assume that the equality \( E_{t-1}^{MP} = E_{t-1}^{CB} \) is true in some period \( t \). This premise holds by definition for \( t=0 \), which serves as the base for our induction argument.

Based on this premise, the comparison of equations (A3) and (A18) immediately yields the inequality \( q_t^{MP} > q_t^{CB} \). We will now use this inequality and the induction premise \( E_{t-1}^{MP} = E_{t-1}^{CB} \) to prove that \( z_t^{MP} > z_t^{CB} \). Assume the contrary: \( z_t^{MP} \leq z_t^{CB} \). Combining this with the above fact \( q_t^{MP} > q_t^{CB} \) and using relationships (A5) and (A32) we obtain the following chain of relationships:

\[
2q_t^{CB} < 2q_t^{MP} = (z_t^{MP})^2 (b + B) A C E_{t-1}^{MP} + z_t^{MP} \frac{2Bh^*}{\theta_t (b + B) - 1} \leq \leq (z_t^{CB})^2 (b + B) A C E_{t-1}^{CB} + z_t^{CB} \frac{2Bh^*}{\theta_t (b + B) - 1} = \frac{\gamma}{\nu(1-\tau)} q_t^{CB} + z_t^{CB} \frac{2Bh^*}{\theta_t (b + B) - 1}
\]

or \( \left(1 - \frac{\gamma}{2\nu(1-\tau)}\right) q_t^{CB} < z_t^{CB} \frac{Bh^*}{\theta_t (b + B) - 1} \). Now using the expressions (A17) and (A18) we rewrite this inequality as

\[
\left(1 - \frac{\gamma}{2\nu(1-\tau)}\right) \nu \tau_t (b + B) A C E_{t-1}^{CB} J_t^{CB} < \left(\frac{\gamma}{2\nu + \gamma} \left(\frac{\tau_t}{1-\tau_t}\right)\right) \left(\frac{J_t^{CB}}{\theta_t (b + B) - 1}\right)^{1/2} \frac{Bh^*}{\theta_t (b + B) - 1}
\]

or equivalently,

\[
\left(\frac{\nu(1-\tau_t)}{\gamma} - \frac{1}{2}\right) \left(\frac{\tau_t}{1-\tau_t}\right)^{1/2} \left(\frac{\gamma}{2\nu + \gamma}\right) \left(\frac{J_t^{CB}}{\theta_t (b + B) - 1}\right)^{1/2} < \frac{Bh^*}{(\theta_t (b + B) - 1) (b + B) A C E_{t-1}^{CB}},
\]

the opposite of the inequality established in (A22). The obtained contradiction thus proves that \( z_t^{MP} > z_t^{CB} \), provided the same prior period’s education quality \( E_{t-1} \). Combined with the earlier
established \( q_i^{MP} > q_i^{CB} \) this implies, according to (8) that, under the same premise, \( E_i^{MP} = E_i^{CB} \).

Segment (b). Observe that both \( q_i^{CB} \) and \( q_i^{MP} \) strictly increase in \( E_{t-1} \). Indeed, according to expressions (A3) and (A18), derivatives of both variables have the sign of the expression

\[
1 - \frac{\theta_i (Bh^*)^2}{\left( \frac{1}{\theta_i} + b \right) \theta_i} = 1 - \frac{a_i^* Bh^*}{A (b + B) ACE_{t-1}}
\]

(A36)

which is positive because \( a_i^* < A \) by definition while \( \frac{Bh^*}{(b + B) ACE_{t-1}} < 1 \) according to Assumption 2 and Lemma 1. Note that inequality \( \frac{Bh^*}{(b + B) ACE_{t-1}} < 1 \) is likewise true under the premise \( E_{t-1}^{MP} = E_{t-1}^{CB} \), and would by far remain valid if \( E_{t-1}^{MP} \geq E_{t-1}^{CB} \) was true. Thus \( \frac{\partial q_i^{CB}}{\partial E_{t-1}^{MP}} > 0 \), \( \frac{\partial q_i^{MP}}{\partial E_{t-1}^{MP}} > 0 \).

Recall that \( \frac{\partial z_i^{CB}}{\partial E_{t-1}^{MP}} > 0 \) according to (A25). In order to prove that \( \frac{\partial z_i^{MP}}{\partial E_{t-1}^{MP}} > 0 \) is also true, we rewrite equation (A6) as \( \tau_i J_i^{MP} = (z_i^{MP})^2 + z_i^{MP} \frac{2Bh^*}{\left( \frac{1}{\theta_i} + b \right) \theta_i} \). The left-hand side of this equation increases in \( E_{t-1}^{MP} \): indeed, \( \frac{\partial J_i^{MP}}{\partial E_{t-1}^{MP}} \) has the sign of the expression

\[
1 - \frac{\theta_i Bh^*}{\left( \frac{1}{\theta_i} + b \right) \theta_i} = 1 - \frac{a_i^{MP}}{A}
\]

Then by differentiating the right-hand side of the equation with respect to \( E_{t-1}^{MP} \) one can easily see that \( \frac{\partial z_i^{MP}}{\partial E_{t-1}^{MP}} > 0 \) must be true.

The established positive effect of \( E_{t-1}^{MP} \) on \( q_i^{MP} \) and \( z_i^{MP} \) implies that the proofs in Segment (a) of the facts \( q_i^{MP} > q_i^{CB} \), \( z_i^{MP} > z_i^{CB} \), and thereby, according to (8), of the fact \( E_{t-1}^{MP} > E_{t-1}^{CB} \) remain valid if the original premise \( E_{t-1}^{MP} = E_{t-1}^{CB} \) is replaced with \( E_{t-1}^{MP} \geq E_{t-1}^{CB} \). This allows us to proceed with the induction argument, which consecutively establishes the inequalities \( q_t^{MP} > q_t^{CB} \), \( z_t^{MP} > z_t^{CB} \), \( E_t^{MP} > E_t^{CB} \) for \( t=0,1,\ldots \), as required. This completes the proof of parts (i) and (ii) of the Theorem.
To prove part (iii), we rewrite the claimed inequality $\tilde{a}_i^{MP} > \tilde{a}_i^{CB}$, using the relationships (15), (6) and (A19), as

$$z_i^{MP} + \frac{\theta_i B^*}{(\theta_i(b+B)-1)ACE_{i-1}^{MP}} > \frac{B^*}{(b+B)ACE_{i-1}^{CB}} + z_i^{CB} \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right)$$

and will use the line of proof similar to that used in Segment (a) in the proof above. Namely, we will initially proceed from the premise, later to be revised, that $E_{i-1}^{MP} = E_{i-1}^{CB}$. Then the above inequality can be rewritten as

$$z_i^{MP} + \frac{B^*}{(\theta_i(b+B)-1)ACE_{i-1}^{MP}} > z_i^{CB} \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right).$$

By squaring both sides of this inequality, we obtain the following equivalent to the claimed inequality:

$$(z_i^{MP})^2 + z_i^{MP} \frac{2B^*}{(b+B)ACE_{i-1}^{MP} (\theta_i(b+B)-1)} + \left( \frac{B^*}{(b+B)(ACE_{i-1}^{MP})^2} \right)^2 > (z_i^{CB})^2 \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right)^2$$

which we can rewrite, according to (A6), (A32), and (A18), as

$$\tau_i J_i^{MP} + \frac{(B^*)^2}{(\theta_i(b+B)-1)^2 (b+B)^2 (ACE_{i-1}^{MP})^2} > \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right)^2 \frac{v(1-\tau_i)}{1-\tau_i} \frac{\gamma}{2\nu + \gamma} J_i^{CB}.$$ 

Thus we have shown that in order to prove the original claim $\tilde{a}_i^{MP} > \tilde{a}_i^{CB}$ it will suffice to establish the following fact $J_i^{MP} \geq \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right)^2 \frac{v(1-\tau_i)}{1-\tau_i} \frac{\gamma}{2\nu + \gamma} J_i^{CB}$. According to the initial premise of $E_{i-1}^{MP} = E_{i-1}^{CB}$ and the expression (A4) we also have $J_i^{MP} = J_i^{CB}$, so we just need to ascertain that

$$1 \geq \left( \frac{v(1-\tau_i)}{\gamma} + \frac{1}{2} \right)^2 \frac{v(1-\tau_i)}{1-\tau_i} \frac{\gamma}{2\nu + \gamma}$$

holds. It is straightforward to show that this is the case iff the parametric condition

$$\frac{\gamma}{2\nu} \geq \frac{(1-\tau_i)}{1+2\sqrt{1-\tau_i}}$$

stated in the Theorem holds.

Thus we have established the inequality $\tilde{a}_i^{MP} > \tilde{a}_i^{CB}$ based on the premise that $E_{i-1}^{MP} = E_{i-1}^{CB}$. It now remains to recall the fact, stated in Lemma 4, that $\tilde{a}_i^{CB}$ decreases in $E_{i-1}^{CB}$, which implies that the fact $\tilde{a}_i^{MP} > \tilde{a}_i^{CB}$ will remain true if the premise $E_{i-1}^{MP} = E_{i-1}^{CB}$ is replaced with $E_{i-1}^{MP} \geq E_{i-1}^{CB}$, the factually correct one as established in part (i) of this Theorem. The proof is thus complete. □
Proof of Theorem 4

Recall the facts \( \frac{\partial q_{MP}^t}{\partial E_{i-1}^t} > 0 \) and \( \frac{\partial z_{MP}^t}{\partial E_{i-1}^t} > 0 \) established in the proof of Theorem 3 for \( t=0,1,\ldots \). They imply, according to (8), that \( \frac{\partial E_{MP}^t}{\partial E_{i-1}^t} > 0 \) is also true at all times. According to Lemma 1, \( E_{0 CB}^t > E_{-1 CB}^t \), while by assumption \( E_{MP}^t = E_{-1 CB}^t \). Combining this fact with the result of part (i) of Theorem 4 we obtain \( E_{0 CB}^t > E_{-1 CB}^t \). Given the positive derivatives stated above, this implies that \( q_{1 MP}^t > q_{0 MP}^t \) and \( z_{1 MP}^t > z_{0 MP}^t \), which by (8) leads to \( E_{1 MP}^t > E_{0 MP}^t \). Continuing this line of argument sequentially, we obtain the Corollary’s result on the growth of the quantity and the absolute quality of teachers, as well as of the quality of education \( E_{i MP}^t \) over time. The latter, according to (6), immediately implies that the college attendance ability cut-off \( \alpha_i^{* MP} \) declines over time.

It remains to prove that \( \alpha_i^{MP} \) also declines over time. With reference to the above argument, it suffices to show that \( \frac{\partial \alpha_i^{MP}}{\partial E_{i-1}^t} < 0 \) for \( t=0,1,\ldots \).

According to (15), \( \alpha_i^{MP} = a_i^* + A z_{i MP}^t \), so

\[
\frac{1}{A} \frac{\partial \alpha_i^{MP}}{\partial E_{i-1}^t} = -\frac{\theta_i B h^*}{(\theta_i (b + B) - 1) A C (E_{i-1}^{MP})} + \frac{\partial z_{i MP}^t}{\partial E_{i-1}^t}
\]

We rewrite equation (A6) as

\[
(z_{i MP}^t)^2 + z_{i MP}^t = \frac{2 B h^*}{(\theta_i (b + B) - 1) (b + B) A C E_{i-1}^{MP}} = \tau_i J_{i MP}
\]

Differentiating both sides and using (6) and (A4) we obtain the following:

\[
\frac{\partial z_{i MP}^t}{\partial E_{i-1}^t} < \frac{B h^*}{(\theta_i (b + B) - 1)(b + B) A C (E_{i-1}^{MP})^2} + \frac{\tau_i B h^*}{2 z_{i MP}^t (b + B) A C (E_{i-1}^{MP})^2} \left( 1 - \frac{a_i^{* MP}}{A} \right)
\]

Substituting this in (A37) we obtain

\[
\frac{1}{A} \frac{\partial \alpha_i^{MP}}{\partial E_{i-1}^t} < \frac{B h^*}{(b + B) A C (E_{i-1}^{MP})^2} \left[ -1 + \frac{\tau_i}{2 z_{i MP}^t} \left( 1 - \frac{a_i^{* MP}}{A} \right) \right]
\]

Now, according to (A26)

\[
\frac{1}{A} \frac{\partial \alpha_i^{CB}}{\partial E_{i-1}^t} = \frac{B h^*}{(b + B) C (E_{CB}^{i-1})^2} \left( -1 + \frac{1}{z_{i CB}^t} \left( 2 \nu (1 - \tau_i) + \gamma \right) \right)
\]
By directly comparing the right-hand sides of this inequality and (A38) and using the facts
\[ z_t^{MP} > z_t^{CB}, \quad E_{t-1}^{MP} \geq E_{t-1}^{CB} \] for \( t=0,1,... \) established in Theorem 4, we conclude that
\[ \frac{\partial a_t^{MP}}{\partial E_{t-1}^{MP}} \leq \frac{\partial a_t^{CB}}{\partial E_{t-1}^{CB}}. \]

Therefore by Lemma 4 \( \frac{\partial a_t^{MP}}{\partial E_{t-1}^{MP}} < 0 \) is true for \( t=0,1,... \). □