Asymptotic approximation of the probability density function of the nonlinear phase noise using the method of steepest descent

V. Vgenopoulou, I. Roudas, K. P. Ho, I. Chochliouros, G. Agapiou, and T. Doukoglou

***Abstract*— Fiber-optic communication systems using phase shift keying (PSK) modulation may suffer from nonlinear phase noise. In this paper, an asymptotic approximation of the probability density function (p.d.f.) of the normalized nonlinear phase noise is derived by taking the inverse Laplace transform of its moment generating function and using the method of steepest descent. For comparison, the inverse Laplace transform of the moment generating function is also numerically evaluated using numerical quadrature. Comparison of the analytical and numerical results, for specific examples, indicates that the method of steepest descent is more accurate and, therefore, is preferable for semi-analytical calculations of the error probability.**

***Index Terms*—Optical fiber communication, fiber nonlinearities, nonlinear phase noise, phase modulation**

I. Introduction

THE error probability is used to characterize the performance of digital lightwave communication systems [1]. Consequently, computationally-efficient algorithms for error probability estimation are of significant importance. Among them, the semi-analytical method for the evaluation of the error probability [1] requires the derivation of an analytical formula for the probability density function (p.d.f.) of the photocurrent at the output of the optical receiver.

Nonlinear phase noise significantly affects the performance of fiber-optic communication systems using different variant of phase shift keying (PSK) modulation [2]. Therefore, the accurate nonlinear phase noise statistics must be taken into account in the calculation of the error probability of these systems [2]. The p.d.f. of the nonlinear phase noise was previously numerically calculated by taking the Fast Fourier Transform (FFT) of its characteristic function, which has been known in closed form [2].

V. Vgenopoulou and I. Roudas are with the Department of Electrical and Computer Engineering, University of Patras, Patras, Greece (corresponding author: I. Roudas, phone:+30 2610 996484; fax:+30 2610 997342; e-mail: roudas@ece.upatras.gr.)

K. P. Ho is with the Institute of Communication Engineering, National Taiwan University, Taipei, Taiwan (e-mail: kpho@ieee.org).

I. Chochliouros, G. Agapiou and T. Doukoglou are with OTE S.A., Athens, Greece (e-mail: {ichochliouros, gagapiou, tdoukoglou} @oteresearch.gr).

In this study, we derive an analytical expression for the p.d.f. of the nonlinear phase noise in a fiber-optic communication system. More specifically, an asymptotic approximation of the p.d.f is calculated by taking the inverse Laplace transform of the nonlinear phase noise moment generating function [3] and by utilizing the method of steepest descent (saddle point approximation) [4]-[7]. For comparison, the inverse Laplace transform of the moment generating function is numerically evaluated using numerical quadrature. It is shown, by example, that the method of steepest descent is faster and more accurate compared to the aforementioned numerical method.

The rest of the paper is organized as follows: Section II initially presents a brief overview of the derivation of the analytical expression for the nonlinear phase noise characteristic function [2]. Then, an asymptotic expression for the p.d.f. of the nonlinear phase noise is calculated by taking the inverse Laplace transform of its moment generating function and using the method of steepest descent. In Section III, the results of the analytical and numerical calculations are compared.

II. Theoretical Model

A. *Nonlinear phase noise characteristic function*

The nonlinear phase noise is due to the interaction of amplified spontaneous emission (ASE) noise generated by the optical amplifiers and the Kerr nonlinearity in the optical fibers. It can be modeled using the following approximate expression [2]

$$\Phi_{NL} = \kappa \int_0^{L_T} \left| E_0 + n(z) \right|^2 dz \qquad (1)$$

where $L_T = NL$ is the total fiber length, N is the number of fiber spans, L is the length of each fiber span, E_0 is the magnitude of the transmitted electric field and κ is the average nonlinear coefficient per unit length. The latter can be written as $\kappa = \gamma L_{eff} / L$, where γ is the fiber's nonlinear coefficient and L_{eff} is the fiber's effective nonlinear length. The term $n(z)$ denotes the total accumulated ASE noise as a function of the distance z and is expressed, in equivalent

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baseband notation [3], as a zero-mean, complex Wiener process [8] with autocorrelation given by $E\{n(z_1)\cdot n^*(z_2)\} = \sigma_s^2 \min(z_1, z_2)$. The term $\sigma_s^2 = 2\sigma_0^2 / L$ is the variance per unit length, and $2\sigma_0^2$ is the ASE noise variance per amplifier per polarization in an optical bandwidth equal to the symbol rate [2]. For mathematical convenience, it is customary to define the normalized nonlinear phase noise as [2]

$$\Phi = \int_0^1 \left|\xi_0 + b(t)\right|^2 dt \tag{2}$$

In (2) the normalized nonlinear phase noise Φ is defined as $\Phi = L\sigma_s^2 \Phi_{NL} / \kappa$. In addition, $t = z/L$ is the normalized distance, $b(t) = n(tL_T)/\sigma_s/\sqrt{L_T}$ is the normalized ASE noise, and $\xi_0 = E_0/\sigma_s/\sqrt{L_T}$ is the magnitude of the normalized transmitted electric field vector. The signal-to-noise ratio (SNR) is defined as $\rho_s = |\xi_0|^2 = |E_0|^2 / L_T\sigma_s^2$.

It is straightforward to show that the corresponding characteristic function is given by [2]

$$\Psi_\phi(iv) = \sec\left(\sqrt{iv}\right)\exp\left[\rho_s\sqrt{iv}\tan\left(\sqrt{iv}\right)\right] \tag{3}$$

If iv in (3) is changed to the complex variable z, we obtain the moment generating function

$$\Psi_\phi(z) = \sec\left(\sqrt{z}\right)\exp\left[\rho_s\sqrt{z}\tan\left(\sqrt{z}\right)\right] \tag{4}$$

The inverse Laplace transform of the moment generating function yields the p.d.f. [9]

$$p(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(-zx)\Psi_\phi(z)\,dz \tag{5}$$

where c denotes a real constant that is suitably chosen to achieve convergence.

B. *Asymptotic approximation of the nonlinear phase noise p.d.f.*

In the following, an asymptotic approximation of the p.d.f. of the normalized nonlinear phase noise is derived. This is achieved by analytically calculating the inverse Laplace transform of its moment generating function, given by (5), by applying the method of steepest descent [4]-[7]. The complex integral can be also evaluated using numerical quadrature (e.g., Simpson's rule or Gauss-Legendre integration) [10].

The method of steepest descent is applied to integrals of the form [5]

$$I(x) = \int_{C'} \exp\left[x\Phi(z)\right]dz \tag{6}$$

where C' is a path in the complex plane, x is a large real positive parameter such that $x \to \infty$, and $\Phi(z) = u(z) + iv(z)$ is an analytic function in the complex plane. The idea of steepest descent is to deform the contour C so that the region of large $u(z)$ is compressed into as short a space as possible [4]. This is equivalent to deform the initial contour C' to a new one C on which $\Phi(z)$ has a constant imaginary part [5]. The paths on which $v(z)$ is constant and the decrease of $u(z)$ is maximal, are called paths of steepest descent. These paths go through saddle points z_0 which are the roots of the equation $\Phi'(z) = 0$ [1]. For such a contour, relation (5) is written

$$p(x) = \frac{1}{2\pi i}\int_C \exp(-zx)\Psi_\phi(z)\,dz \tag{7}$$

or equivalently

$$p(x) = \frac{1}{2\pi i}\int_C \exp\left\{x\left[-z + \frac{1}{x}\ln\Psi_\phi(z)\right]\right\}dz \tag{8}$$

In our case, we observe that the auxiliary function $\Phi(z) = -z + \ln\Psi_\phi(z)/x$. Then (8) is written as

$$p(x) = \frac{1}{2\pi i}\int_C \exp\left[x\Phi(z)\right]dz \tag{9}$$

The saddle point z_0 is calculated by taking the first derivative of $\Phi(z)$ [5]

$$\Phi'(z) = -1 + \frac{1}{x}\frac{\Psi'_\phi(z)}{\Psi_\phi(z)} \tag{10}$$

and setting $\Phi'(z)$ to zero

$$\frac{\Psi'_\phi(z_0)}{\Psi_\phi(z_0)} = x \tag{11}$$

Equation (11) is nonlinear and therefore, must be computed numerically using an iterative procedure, e.g., the bracketing method [10].

In Fig. 1, the function $f(z) = \Psi'_\phi(z)/\Psi_\phi(z)$ is depicted together with the function $g(z) = x$ for three different values of x. We observe that $f(z)$ goes to infinity for $z = \left[(2\kappa+1)\pi/2\right]^2$, where $\kappa = 1, 2\ldots$. The points where $f(z)$, $g(z)$ intersect correspond to the saddle points z_0. The saddle corresponding to the leftmost branch of the function $f(z) = \Psi'_\phi(z)/\Psi_\phi(z)$, which lies in the interval $(-\infty, \pi^2/4)$, is the desired root because its contribution to the value of the integrand (7) dominates the contribution of all other saddle points.

[1] In the following primes denote derivatives of different orders with respect to z.

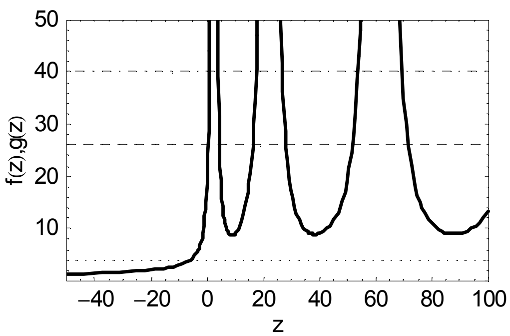

Fig.1 Graphical solution of (11), showing plots of $f(z) = \Psi'_\phi(z)/\Psi_\phi(z)$ and $g(z) = x$, for three different values of x. (Symbols: $f(z)$: solid line, x =4: dotted line, x =26: dashed line and x =40: dashed-dotted line).

An approximate analytical expression of the leftmost saddle point in Fig. 1, for small values of x, can be derived by arbitrarily assuming that the nonlinear phase noise statistics can be fairly well approximated by a Gaussian distribution. In this case, the characteristic function is written as [3]

$$\Psi_{\text{approx}}(i\upsilon) = \exp\left(i\upsilon\mu - \frac{\upsilon^2\sigma^2}{2}\right) \tag{12}$$

where μ, σ^2 denote the mean and the variance of the nonlinear phase noise, respectively. The latter are determined by taking the first and second derivative, respectively, of the exact characteristic function (3) with respect to υ and setting $\upsilon = 0$ [3]

$$\mu = \rho_s + \frac{1}{2} \tag{13}$$

$$\sigma^2 = \frac{2}{3}\rho_s + \frac{1}{6} \tag{14}$$

Setting $i\upsilon = z$ and substituting (12) into (8) yields

$$p_{\text{approx}}(x) = \frac{1}{2\pi i}\int_C \exp\left(-zx + z\mu + \frac{z^2\sigma^2}{2}\right)dx \tag{15}$$

The auxiliary function becomes

$$\Phi_{\text{approx}}(z) = -z + \frac{z}{x}\mu + \frac{z^2\sigma^2}{2x} \tag{16}$$

The approximate value for the leftmost saddle point z_0 in Fig. 1 is calculated by evaluating the root of the first derivative of (16) with respect to z. It is straightforward to show that

$$z_0^{(\text{approx})} = \frac{x-\mu}{\sigma^2} \tag{17}$$

The approximate value $z_0^{(\text{approx})}$ can be used as a starting point for the iterative procedure used to solve (11). However, it should be stressed that this approximation is useful only for small values of x. As x increases, $z_0^{(\text{approx})}$ increases and eventually exceeds $\pi^2/4$, which is the upper bound of the interval where the desired saddle point lies.

Once the saddle point z_0 is known, we expand $\Phi(z)$ in Taylor series around z_0

$$\Phi(z) = \Phi(z_0) + \frac{1}{2}(z-z_0)^2\Phi''(z_0) + \mathcal{O}(z^3) \tag{18}$$

where

$$\Phi''(z_0) = \frac{1}{x}\left[\frac{\Psi''_\phi(z_0)}{\Psi_\phi(z_0)} - x^2\right] \tag{19}$$

The path of steepest descent lies along a straight line paralle to the imaginary axis passing from z_0 since $\upsilon(z_0)$ along this path is $\upsilon(z) = \upsilon(z_0) = 0$. Substituting (18) and (19) into (9) and setting $z = z_0 + i\omega \Rightarrow dz = id\omega$ yields

$$\begin{aligned} p(x) \sim & \frac{1}{2\pi}\exp(-xz_0)\Psi_\phi(z_0) \\ & \cdot\int_{-\infty}^{\infty}\exp\left\{\left(-\frac{\omega^2}{2}\right)\left[x\Phi''(z_0)\right]\right\}d\omega \end{aligned} \tag{20}$$

or equivalently

$$p(x) \sim \frac{\exp(-xz_0)\Psi_\phi(z_0)}{\sqrt{2\pi x\Phi''(z_0)}} \tag{21}$$

Expression (21) is the asymptotic approximation of the p.d.f. of the normalized nonlinear phase noise for $x \to \infty$. In Section III, we test the accuracy of (21) for a given range of values for x.

III. Results and Discussion

In this section, to illustrate the model, we plot the p.d.f. of the normalized nonlinear phase noise calculated both numerically and analytically, for three different values of the SNR ρ_s.

In Fig. 2, the numerical quadrature and the asymptotic approximation (21) of the p.d.f. of the normalized nonlinear phase noise are plotted in linear scale, for ρ_s=11, 18 and 24. The accurate (solid line) and the approximate (dashed line) p.d.f. cannot be distinguished in the scale of the graph. Moreover, it is observed that for increasing SNR, the p.d.f. curves are shifted to the right and become wider, in agreement with (13) and (14).

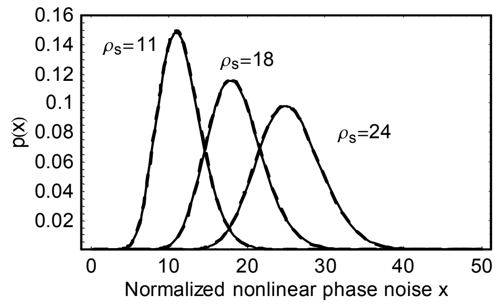

Fig. 2 Probability density function of the normalized nonlinear phase noise given by numerical quadrature (solid line) and asymptotic approximation (dashed line) in linear scale for SNR ρ_s =11, 18 and 24.

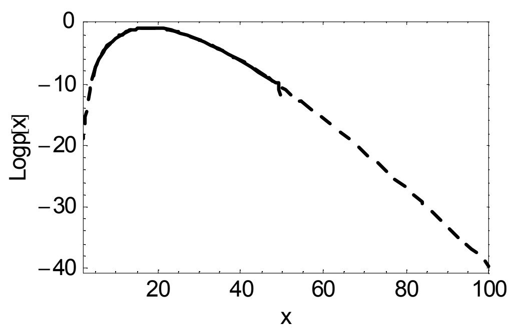

Fig. 3 Probability density function of the normalized nonlinear phase noise given by numerical quadrature (solid line) and asymptotic approximation (dashed line) in logarithmic scale for SNR ρ_s=18.

In Fig. 3, the solid line corresponds to the numerical quadrature approximation of the p.d.f.. The dashed line depicts the asymptotic approximation of the p.d.f. (21). Both graphs are plotted in logarithmic scale for $\rho_s = 18$. The logarithmic scale is used in order to highlight the difference between the tails of the p.d.f.s evaluated using numerical quadrature and the asymptotic approximation. The p.d.f. tail evaluated using the numerical quadrature reaches a floor around 10^{-10} due to numerical errors, while the tail of the asymptotic approximation of the p.d.f. provides accuracy of at least 20 significant digits and therefore it is more suitable for small-error probability calculations.

IV. Conclusions

We used the method of steepest descent in order to derive an asymptotic approximation for the p.d.f. of the normalized nonlinear phase noise from its moment generating function. The accuracy of the asymptotic approximation of the p.d.f. is by far superior to numerical quadrature. Most importantly, it is very precise at the p.d.f. tails from where the error probability can be determined. Therefore, it should be the method of choice for semi-analytical evaluations of low error probabilities for fiber-optic communication systems using PSK modulation.

Finally, it should be emphasized that the expression of the characteristic function of the nonlinear phase noise used as a starting point for our calculation contains several idealizations [11] and is valid for a large number of fiber spans exclusively [2]. Nevertheless, it is anticipated that the method of steepest descent should be applicable for alternative nonlinear phase noise models as well. The study of these alternative models does not lie within the scope of the present paper and will be part of future work.

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