Numerical resolution of the Fokker-Planck equation for the study of phase noise filtering in coherent optical systems

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ABSTRACT

Laser phase noise deteriorates the high sensitivity of heterodyne optical receivers. To reduce phase noise influence, the intermediate frequency (IF) signal resulting from the coherent detection is filtered by a narrow bandpass filter (BPF). The phase noise at the input of the BPF generates an amplitude and phase noise at the output of the BPF. The joint probability density function of these noises is evaluated in the case of a first order (RC) filter by numerical resolution of a Fokker-Planck equation. A finite difference operator splitting scheme is used. The accuracy of the numerical solution is checked comparing numerically and analytically calculated moments. In addition, a new very efficient method for the analytical calculation of moments is developed.

Contour plots of the probability density for both a finite time integrator and a first order filter are compared in order to show the impact of different filter types on phase noise filtering. The marginal pdf of the amplitude and phase noise at the output of the above filters are also calculated.

Keywords : Coherent optical systems, phase noise filtering, Fokker-Planck equation.

1 INTRODUCTION

Coherent optical communication systems are attractive candidates for OFDM (Optical Frequency Division Multiplexing) networks because of their increased selectivity in comparison with direct detection schemes. In such networks, lasers with significant phase noise may be used in order to reduce the cost of the implementation. For this reason, it is interesting to study the impact of the IF filtering on phase noise in coherent optical communication systems.

Several papers have been published on this subject (see for example the article by Garrett et al.[1] and its references). It was shown that IF filtering reduces the phase noise variance but also generates an amplitude noise at the output of the IF filter. The joint probability density function of the amplitude and phase noise at the output of the IF filter can be obtained by resolution of a Fokker-Planck equation.[2]

This equation has been resolved numerically[1,3,4] and, more recently, analytically[5] for a finite time integrator filter (or I&D, for *Integrate & Dump*). However, the finite time integrator is a mathematical model that does not have any physical counterpart. Therefore, the numerical method proposed by Waite and Lettis[4] is generalized

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here for a first order filter, which enables a more realistic analysis of the system performance.

The remainder of the paper is organized as follows : The second section presents the mathematical model which leads to a Fokker-Planck equation for a first order filter; the third section is devoted to the numerical resolution of this equation and the development of an initial condition useful for the numerical calculations; the fourth part presents a new, very efficient method for the evaluation of the theoretical power moments. The moments are used to check the accuracy of the numerical solution. The two last sections present the results and the conclusions respectively.

2 THEORY

A block diagram of a heterodyne optical receiver is given in fig. 1. The optical signal from the transmitter is combined with the signal of the local oscillator on a PIN photodiode. The resulting photocurrent is amplified and filtered by a narrow bandpass filter (BPF) in order to select the desired channel, reject the direct detection component, and reduce shot and phase noise.

The intermediate frequency (IF) component of the photocurrent can be written as :

$$i_{ph}(t) = \Re\{\tilde{i}_{ph}(t)e^{i\omega_{IF}t}\} \tag{1}$$

where $\omega_{IF} = 2\pi(\nu_s - \nu_l)$ is the angular intermediate frequency (IF) and $\tilde{i}_{ph}(t)$ is the complex envelope, which, in the absence of modulation, can be expressed as :

$$\tilde{i}_{ph}(t) = Ae^{i\phi(t)} + \tilde{n}(t) \tag{2}$$

where $A = 2R\sqrt{P_s P_l}$ is the amplitude of the IF photocurrent, P_s is the received average optical power, P_l is the average optical power of the local oscillator, R is the photodiode sensitivity, $\phi(t)$ is the IF phase noise, and $\tilde{n}(t)$ is the complex envelope of the shot and thermal noise.

Phase noise is due to the spontaneous emission in semiconductor lasers. It can be modelled by the following integral equation

$$\phi(t) = \int_0^t \dot{\phi}(\tau)\, d\tau \tag{3}$$

where $\dot{\phi}(t)$ represents the instantaneous angular frequency fluctuations.

It is customary to model $\dot{\phi}(t)$ as a white Gaussian noise with zero mean and two-sided power spectral density equal to

$$S_{\dot{\phi}}(\omega) = D = 2\pi\Delta\nu \tag{4}$$

where D is the phase diffusion coefficient and $\Delta\nu$ is the full width at half maximum (FWHM) of the non-modulated optical field at the intermediate frequency (IF).

The above assumptions for $\dot{\phi}(t)$ make the phase noise defined by (3) a *Wiener process*. The properties of the Wiener process are well known[6,7] :

i) $\phi(t)$ is Gaussian, with zero mean and autocorrelation :

$$R_\phi(t_1, t_2) = D\min\{t_1, t_2\} \tag{5}$$

ii) $\phi(t)$ has independent increments in nonoverlaping time intervals, i.e. $\phi(t_2) - \phi(t_1)$ and $\phi(t_4) - \phi(t_3)$ are independent if $t_4 > t_3 > t_2 > t_1$.

iii) $\phi(t)$ has stationary increments, i.e. the probability density of $\Delta\phi = \phi(t_1) - \phi(t_2)$ is Gaussian with zero mean and variance $D(t_1 - t_2)$.

The mathematical definition of the problem that we are going to address here is as follows : if $\tilde{h}(t)$ is the equivalent lowpass impulse response of the IF filter, we want to calculate the probability density function (pdf) of the complex envelope $\tilde{z}(t)$ at the IF filter output :

$$\tilde{z}(t) = r(t)e^{i\theta(t)} = \int_0^\infty \tilde{h}(t-t')\,\tilde{i}_{ph}(t')dt' = \int_0^\infty \tilde{h}(t-t')e^{i\phi(t')}dt' \tag{6}$$

In the above expression, the shot and thermal noise terms are omitted since we are interested uniquely in the study of the phase noise filtering.

The stochastic process $\tilde{z}(t)$ is shown to be a *diffusion process.*[2] The transition probability density $p(r,\theta,t)$ of the diffusion processes from a state $\tilde{z}_0(t_0) = (r_0,\theta_0;t_0)$ to a state $\tilde{z}(t) = (r,\theta;t)$, can be calculated by resolution of partial differential equations called *Fokker-Planck equations.*[8,9]

Bond[2] has shown that the Fokker-Planck equation corresponding to (6) is given by:

$$\frac{D}{2}\frac{\partial^2}{\partial\theta^2}p(r,\theta,t) - \cos\theta\tilde{h}(t)\frac{\partial}{\partial r}p(r,\theta,t) + \frac{1}{r}\sin\theta\tilde{h}(t)\frac{\partial}{\partial\theta}p(r,\theta,t) = \frac{\partial}{\partial t}p(r,\theta,t) \tag{7}$$

satisfying the initial condition

$$p(r,\theta,0) = \delta(r,\theta) \tag{8}$$

The equivalent lowpass impulse response of a first order filter is written as

$$\tilde{h}(t) = \begin{cases} ce^{-at} & 0 \le t \le \tau_1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

In (9) we have implicitly assumed that $\tilde{h}(t)$ is truncated in time, in order to avoid intersymbol interference and noise correlation.

The normalization constant c is calculated in order to obtain $\int_{-\infty}^{\infty}\tilde{h}(t)dt = 1$. Consequently

$$c = \frac{a}{1 - e^{-a\tau_1}} \tag{10}$$

It is worth noting from (9), (10) that when $a \to 0$, the first order filter becomes equivalent to the finite time integrator.

The equivalent noise bandwidth of the first order filter is given by

$$B_{eq} \triangleq \frac{\int_{-\infty}^{\infty}|H(f)|^2df}{max|H(f)|^2} = \frac{a}{2}\coth\frac{a\tau_1}{2} \tag{11}$$

Fig. 2 shows the equivalent noise bandwidth B_{eq} normalized to the filter impulse response duration τ_1 as a function of $a\tau_1/2$. It is worth noting that for small $a\tau_1$ the equivalent noise bandwidth tends to $1/\tau_1$. By increasing the value of $a\tau_1$ the equivalent noise bandwidth becomes almost equal to $a/2$.

By substituting (9) in (7), we obtain:

$$\frac{D}{2}\frac{\partial^2}{\partial\theta^2}p(r,\theta,t) - c\cos\theta e^{-at}\frac{\partial}{\partial r}p(r,\theta,t) + \frac{c\sin\theta}{r}e^{-at}\frac{\partial}{\partial\theta}p(r,\theta,t) = \frac{\partial}{\partial t}p(r,\theta,t) \tag{12}$$

It is convenient to write (12) in dimensionless form by setting $t' = t/\tau_1$, $c' = c\tau_1$, $a' = a\tau_1$, $D' = D\tau_1$. The equation (12) is written :

$$\frac{D'}{2}\frac{\partial^2}{\partial\theta^2}p(r,\theta,t') - c'e^{-a't'}\cos\theta\frac{\partial}{\partial r}p(r,\theta,t') + c'e^{-a't'}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}p(r,\theta,t') = \frac{\partial}{\partial t'}p(r,\theta,t') \tag{13}$$

In the following, for the sake of clarity, we will drop the primes and use the notation a, c, D, $p(r,\theta,t)$ instead of a', c', D', $p(r,\theta,t')$.

Finally, by substituting $P = rp$, the equation (13) is reduced to the more compact form of a conservation law:

$$-\frac{\partial}{\partial r}(ce^{-at}P\cos\theta) + \frac{\partial}{\partial\theta}(ce^{-at}\frac{P}{r}\sin\theta + \frac{D}{2}\frac{\partial P}{\partial\theta}) = \frac{\partial P}{\partial t} \tag{14}$$

satisfying the initial condition (8).

3 NUMERICAL RESOLUTION

In order to resolve numerically the above equation, the finite-difference scheme by Waite and Lettis[4] is used. The algorithm is based on a finite difference method called operator splitting.[10] Equation (14) is divided into two sub-equations, each one containing variations only in the radial or the angular direction. The principle of the method is the following: during the first half of each timestep, the radial sub-equation is solved using a non-linear explicit scheme which is highly accurate and free of numerical oscillations.[11] During the second half of each timestep, the angular sub-equation is solved using an implicit scheme.

3.1 Initial condition

Rather than using (8), which is difficult to approximate numerically, an approximation to the real solution for small times is derived and used as initial condition.

By developing the exponential in (6) in Taylor series we obtain for small times ($t \to 0$)

$$r(t) \simeq \frac{ct}{\tau_1} \tag{15a}$$

$$\theta(t) \simeq \int_0^t \phi(t')dt' - a\int_0^t t'\phi(t')dt' \tag{15b}$$

From (15a) we note that the initial distribution $P(r,\theta,t)$ is concentrated on the circle $r \simeq ct/\tau_1$.

According to Foschini et Vannucci[12] the integrals of (15b) are decomposed in the ortho-normal base $X_n = \sqrt{2}\int \sin n\pi\tau d\phi$. It is straightforward to show that

$$\theta(t) = \sqrt{Dt}\sum_{n=1}^{\infty}\left[\frac{\sqrt{2}}{n\pi}\left(1+\frac{at}{2}\right) + at\frac{2\sqrt{2}}{(n\pi)^3}l_0\right]X_n = \sum_{n=1}^{\infty}K_nX_n = \mathbf{KX} \tag{16}$$

where l_0 is the odd term indicator sequence (i.e. $l_0 = 1,0,1,0\ldots$), and $\mathbf{K}$, $\mathbf{X}$ are arrays with elements K_n, X_n respectively.

The moment generating function of $\theta(t)$ can be written in terms of the Laplace transform

$$L = E\left\{e^{s\theta(t)}\right\} = e^{\frac{s^2||K||^2}{2}} \tag{17}$$

The initial marginal phase distribution $p(\theta,t)$ can be calculated by the inversion of the moment generating function and is shown to be Gaussian with variance

$$\sigma_\theta^2 = \left.\frac{\partial L}{\partial s^2}\right|_{s=0} = ||K||^2 = \left(\frac{1}{3} + \frac{5}{12}at + \frac{2}{15}a^2t^2\right) Dt \tag{18}$$

4 THEORETICAL MOMENTS

4.1 General formalism

To check the accuracy of the numerical solution, it is necessary to define a criterion. Garrett et al.[1] compared numerically and analytically calculated moments for a finite time integrator filter. Analytical moments are calculated by a recursive expression developed by Bond.[2]

Here, we develop a new method for the analytical calculation of moments that presents two advantages : (i) it has an almost negligible computational complexity allowing to obtain moments of much higher order than previous methods[2,13]; (ii) it is applicable to a wide class of filter types.

The power moments $\mu_n(t) = E\{\tilde{z}(t)^n\}$ are defined by

$$\mu_n(t) \triangleq E\left\{\left(\int_0^\infty \tilde{h}(t-s)e^{i\phi(s)}\,ds\right)^n\right\} = E\left\{\int_0^\infty \cdots \int_0^\infty \tilde{h}(t-s_1)\cdots\tilde{h}(t-s_n)\, e^{i\sum_{k=1}^{n}\phi(s_k)}\, ds_1\cdots ds_n\right\} \tag{19}$$

Note the symmetry of the integrand with respect to a permutation of the integration variables $s_1,\ldots,s_n$. This symmetry can be exploited to impose a particular integration order in (19), i.e. $s_1 > s_2 > s_3 > \cdots > s_{n-1} > s_n$, in order to obtain

$$\mu_n(t) = E\left\{n!\int_0^\infty\int_0^{s_1}\cdots\int_0^{s_{n-1}} \tilde{h}(t-s_1)\cdots\tilde{h}(t-s_n)\, e^{i\sum_{k=1}^{n}\phi(s_k)}\, ds_n\cdots ds_1\right\} \tag{20}$$

As $\tilde{h}(\cdot)$ is a deterministic function, the expectation is taken over the exponential of the integrand in (20) :

$$\mu_n(t) = n!\int_0^\infty\int_0^{s_1}\cdots\int_0^{s_{n-1}} \tilde{h}(t-s_1)\cdots\tilde{h}(t-s_n)\, E\left\{e^{i\sum_{k=1}^{n}\phi(s_k)}\right\} ds_n\cdots ds_1 \tag{21}$$

The sum in (21) can be written in the form:

$$\sum_{k=1}^{n}\phi(s_k) = \Big[\phi(s_1)-\phi(s_2)\Big] + 2\Big[\phi(s_2)-\phi(s_3)\Big] + \cdots + (n-1)\Big[\phi(s_{n-1})-\phi(s_n)\Big] + n\Big[\phi(s_n)-0\Big]$$

From the properties (ii), (iii) of the Wiener process. the phase differences in the brackets $\Delta\phi_k = [\phi(s_k) - \phi(s_{k+1})]$ are independent Gaussian processes with zero mean and variances $\sigma_k^2 = D(s_k - s_{k+1})$. The expectation of (21) is now easily evaluated by[8]

$$E\left\{ e^{\,i\sum_{k=1}^{n}\phi(s_k)} \right\} = \exp\Big[\xi_1(s_1 - s_2) + \xi_2(s_2 - s_3) + \cdots + \xi_{n-1}(s_{n-1} - s_n) + \xi_n(s_n - 0)\Big] \tag{22}$$

where we used the shorthand notation $\xi_k = -D\,k^2/2$.

By substituting (22) in (21), and after some rearrangement, we obtain the final expression for the power moments, up to now independent of the filter impulse response, as

$$\mu_n(t) = n! \int_0^\infty ds_n\, \tilde{h}(t - s_n)\, e^{\xi_n s_n} \int_{s_n}^\infty ds_{n-1}\, \tilde{h}(t - s_{n-1})\, e^{\xi_{n-1}(s_{n-1} - s_n)} \cdots \int_{s_2}^\infty ds_1\, \tilde{h}(t - s_1)\, e^{\xi_1(s_1 - s_2)} \tag{23}$$

4.2 First order filter

By substituting (9) in (23), we obtain

$$\mu_n(t) = n!\, c^n\, e^{-nat} \int_0^t ds_n\, e^{(\xi_n + na)s_n} \int_{s_n}^t ds_{n-1}\, e^{(\xi_{n-1} + (n-1)a)(s_{n-1} - s_n)} \cdots \int_{s_2}^t ds_1\, e^{(\xi_1 + a)(s_1 - s_2)} \tag{24}$$

This form is similar to the kernel of a Laplace transform. Taking the transform $L = \int_0^\infty e^{-\gamma t}\mu_n(t)\,dt$ and changing the integration order, we obtain :

$$L = n!\, c^n \int_0^\infty ds_n\, e^{(\xi_n + na)s_n} \int_{s_n}^\infty ds_{n-1}\, e^{(\xi_{n-1} + (n-1)a)(s_{n-1} - s_n)} \cdots \int_{s_2}^\infty ds_1\, e^{(\xi_1 + a)(s_1 - s_2)} \int_{s_1}^\infty dt\, e^{(-na - \gamma)t}$$

We separate the variables :

$$L = n!\, c^n \int_0^\infty ds_n\, e^{(\xi_n - \gamma)s_n} \int_{s_n}^\infty ds_{n-1}\, e^{(\xi_{n-1} - a - \gamma)(s_{n-1} - s_n)} \cdots \int_{s_2}^\infty ds_1\, e^{(\xi_1 - (n-1)a - \gamma)(s_1 - s_2)} \int_{s_1}^\infty dt\, e^{(-na - \gamma)(t - s_1)}$$

and calculate each integral separately :

$$\begin{aligned} L &= \frac{n!\, c^n}{\prod_{j=0}^{n}\left[\gamma + (n-j)a - \xi_j\right]} = n!\, c^n \sum_{j=0}^{n} \frac{c_j}{\gamma + (n-j)a - \xi_j} \\ &= n!\, c^n \sum_{j=0}^{n} c_j \int_0^\infty e^{-\gamma t} e^{((n-j) - a + \xi_j)t}\, dt = \int_0^\infty \underbrace{n!\, c^n \sum_{j=0}^{n} c_j e^{((n-j) - a + \xi_j)t}}_{\mu_n(t)}\, e^{-\gamma t} dt \end{aligned}$$

The coefficients c_j are defined by

$$c_j^{-1} = \prod_{\substack{k=0 \\ k \neq j}}^{n} \Big[-(k-j)a + \xi_j - \xi_k \Big]. \tag{25}$$

By substituting $\xi_r = -D\,r^2/2$, we obtain the final expression for the power moments at the output of the first order filter :

$$E\{z(t)^n\} = n!\,c^n \sum_{j=0}^{n} c_j \exp\left[-(n-j)at - \frac{j^2}{2}Dt\right] \tag{26}$$

with

$$c_j^{-1} = (-1)^j\, j!\,(n-j)! \prod_{\substack{k=0 \\ k\neq j}}^{n} \left[-a + \frac{j+k}{2}D\right] \tag{27}$$

For the finite time integrator filter, $a = 0$ and the expressions (26), (27) are written

$$E\{z(t)^n\} = n!\,c^n \sum_{j=0}^{n} c_j \exp\left[-\frac{j^2}{2}Dt\right] \tag{28}$$

with

$$c_0 = \frac{2^n D^{-n}}{(n!)^2} \tag{29a}$$

$$c_j = \frac{(-1)^j\, 2^{n+1} D^{-n}}{(n-j)!\,(n+j)!} \tag{29b}$$

The five first power moments for a first order filter are shown below (where we introduced the shorthand notation $[x] \triangleq (-a + \frac{D}{2}x)/c$ and $d \triangleq D/2$)

$$E\{z\} = \frac{c}{-a+d}\left(e^{-at} - e^{-dt}\right) \qquad E\{z^2\} = \frac{e^{-2at}}{[1][2]} - \frac{2e^{(-a-d)t}}{[1][3]} + \frac{e^{-4dt}}{[2][3]}$$

$$E\{z^3\} = \frac{e^{-3t}}{[1][2][3]} - \frac{3e^{(-2a-d)t}}{[1][3][4]} + \frac{3e^{(-a-4d)t}}{[2][3][5]} - \frac{e^{-9dt}}{[3][4][5]}$$

$$E\{z^4\} = \frac{e^{-4at}}{[1][2][3][4]} - \frac{4e^{(-3a-d)t}}{[1][3][4][5]} + \frac{6e^{(-2a-4d)t}}{[2][3][5][6]} - \frac{4e^{(-a-9d)t}}{[3][4][5][7]} + \frac{e^{-16dt}}{[4][5][6][7]}$$

$$E\{z^5\} = \frac{e^{-5at}}{[1][2][3][4][5]} - \frac{5e^{(-4a-d)t}}{[1][3][4][5][6]} + \frac{10e^{(-3a-4d)t}}{[2][3][5][6][7]} - \frac{10e^{(-2a-9d)t}}{[3][4][5][7][8]} + \frac{5e^{(-a-16d)t}}{[4][5][6][7][9]} - \frac{e^{-25dt}}{[5][6][7][8][9]}$$

We also compared the moments at the output of the finite time integrator filter as given by (28) with those obtained by Bond's expression[2]

$$\mu_n(t) = \exp\left(-n^2 Dt/2\right) \int_0^t n\,\mu_{n-1}(\tau) \exp\left(n^2 D\tau/2\right) d\tau \tag{30}$$

satisfying the initial condition $\mu_0(t) = 1$.

Our method is clearly more efficient, as depicted in fig. 3.

5 RESULTS

In the following, the pdf $P(r,\theta,t)$ is calculated numerically, for two types of filters : (i) a finite time integrator; (ii) a first order filter.

In order to do a meaningful comparison, it is assumed that the same phase noise (i.e. characterized by the same diffusion coefficient D) is applied at the input of both filters. In addition, both filters have the same equivalent noise bandwidth. In the following, the equivalent noise bandwidth of the finite time integrator filter is denoted by B_{eq}^{FI} and the equivalent noise bandwidth of the RC filter by B_{eq}^{RC}. We will examine the special case when $D\tau_1^{FI} = 1$, $\tau_1^{RC}/\tau_1^{FI} = 3$. From (11), it is straightforward to see that $a^{RC}\tau_1^{RC} = 5.97$ in order to have $B_{eq}^{FI} = B_{eq}^{RC}$.

Fig. 4 shows contour plots of $P(r,\theta,t)$ at the instant $t = \tau_1^{FI}$ for a finite time integrator filter. Contours (a)-(d) correspond to values of P equal to 0.05, 0.1, 0.5,1. At small times, all the pdf is concentrated on a circle around the origin. At $t = \tau_1^{FI}$, the contours have propagated along the direction of $\theta = 0$ and the maximum of P is located at about $r = 0.95$. Note that the contours are narrower in the direction $\theta = 0$ and become wider as θ increases. The reason for this phenomenon is that the advection velocity is greater along the direction $\theta = 0$ than in the other directions, so the circular contours are deformed during their propagation.

Fig. 5 shows contour plots of $P(r,\theta,t)$ at the instant $t = \tau_1^{RC}$ for a finite time integrator. In contrast with the contours of fig. 4, the contours present almost a circular symmetry around the origin. On the other hand, the angular diffusion is larger for the RC filter under study due to the fact that $D\tau_1^{RC} = 3$. For this reason the pdf is less concentrated in the angular direction and the attenuation of the pdf peak is larger.

The amplitude noise resulting from the conversion phase-amplitude due to the IF filtering is almost the same for both filter types (fig. 6). This justifies the statement of Foschini and Vannucci[12] that the phase noise filtering is not affected in a fundamental way by the filter shape but only by the filter equivalent noise bandwidth.

A probability distribution that presents a particular interest for phase and frequency modulation schemes is the marginal phase noise pdf $p_\theta(\theta,t)$ at the output of the IF filter. Fig. 7 shows $p_\theta(\theta,t)$ as a function of θ for the two different filter types. We observe that the phase error is much larger in the case of an RC filter. This is due to the fact that the statistics of the phase noise after IF filtering are determined by the product $D\tau_1$ rather than the ratio D/B_{eq}. As in this particular example we assumed that $D\tau_1^{FI} = 1$ and $D\tau_1^{RC} = 3$, it is clear that phase noise is enhanced in the case of an RC filter. However, it is difficult to conclude whether the finite time integrator filter is more efficient in phase noise filtering than an RC filter. For example, in a differential receiver, the linewidth-delay line product $\Delta\nu\tau$ determines the system performance. Therefore, one has to examine whether it is better to filter the phase noise with a finite time integrator filter having an impulse response duration τ_1^{FI} less than the delay τ of the differential demodulator, or it is better to use an RC filter having the same equivalent noise bandwidth as the finite time integrator filter, but an impulse response duration τ_1^{RC} equal to the delay τ of the differential demodulator. This is an interesting subject for further study.

6 CONCLUSION

This paper accurately evaluates the joint probability distribution of the amplitude and phase noise resulting from the IF filtering of phase noise by a first order (RC) filter. The study is based on the numerical resolution of a Fokker-Planck equation.

A new method for the analytical evaluation of the power moments at the output of the IF filter is also developed. This method is applicable to a wide class of filters and presents a very low computational complexity.

Here it is used to check the accuracy of the numerical results.

Finally, a particular example is given in order to illustrate the impact of a finite time integrator and a first order filter on phase noise.

The study of the phase noise filtering by means of the Fokker-Planck equation is a powerful and promising method and it can be generalized in the future for other filter types and modulation schemes (i.e. CPFSK).

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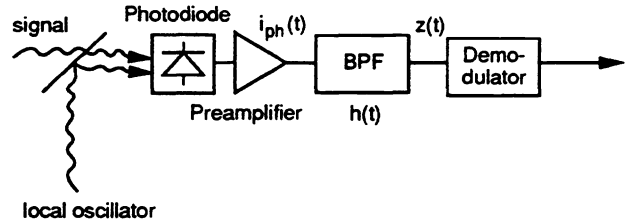

Figure 1: Block diagram of a heterodyne optical receiver.

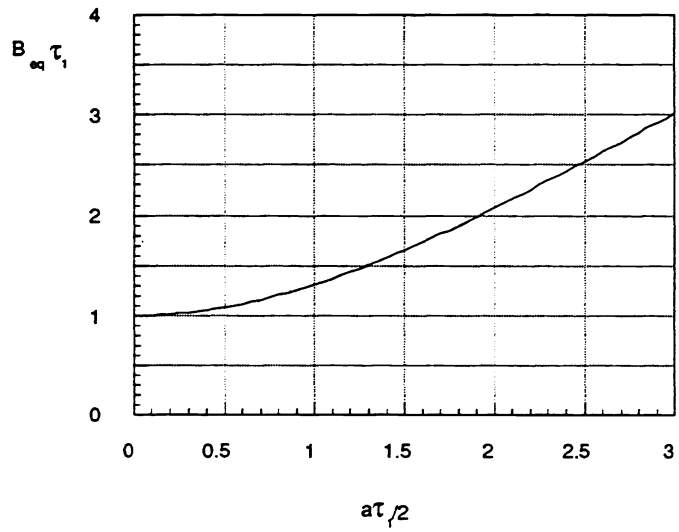

Figure 2: Equivalent noise bandwidth normalized to the filter impulse response duration τ_1 as a function of $a\tau_1/2$.

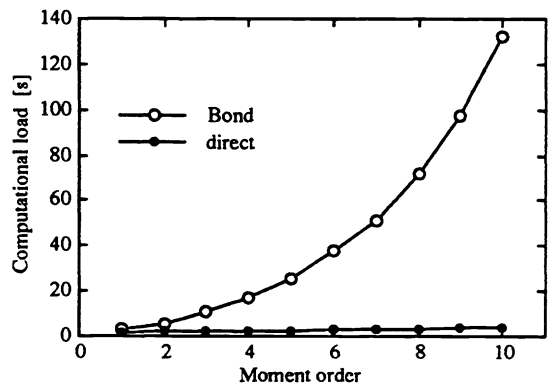

Figure 3: Computing time (in seconds) with Bond's method and our approach. Computations were performed by Mathematica[14] on a Macintosh Quadra 700.

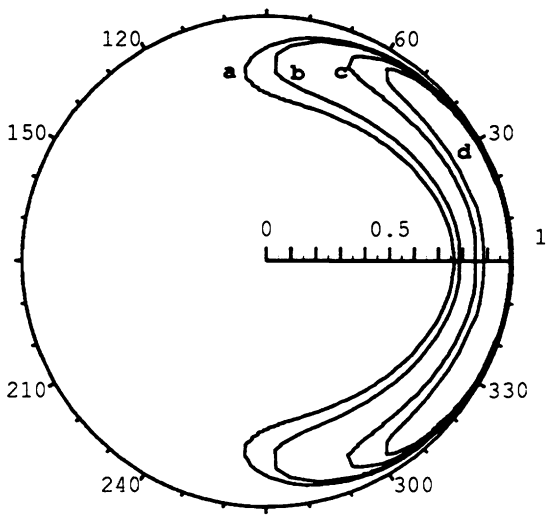

Figure 4: Contour plot of $P(r,\theta,t)$ at the instant $t = \tau_1^{FI}$ for a finite time integrator filter. Contours (a)-(d) correspond to values of P equal to 0.05, 0.1, 0.5,1 respectively. (Conditions : $D\tau_1^{FI} = 1$, grid 1000 x 500).

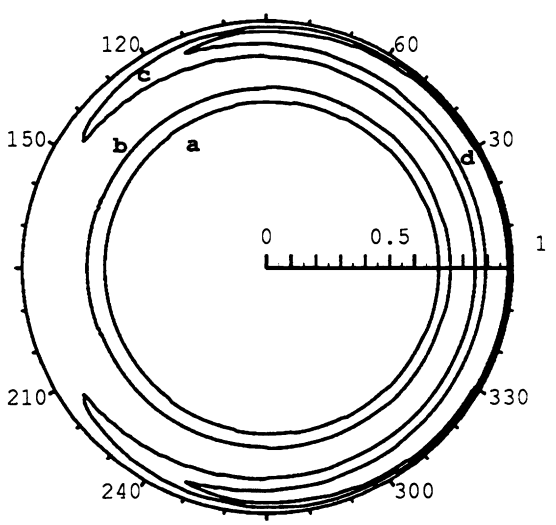

Figure 5: Same as previously for a first order (RC) filter (Conditions : $D\tau_1^{RC} = 3$. $a\tau_1^{RC} = 5.97$, grid 1000 x 500).

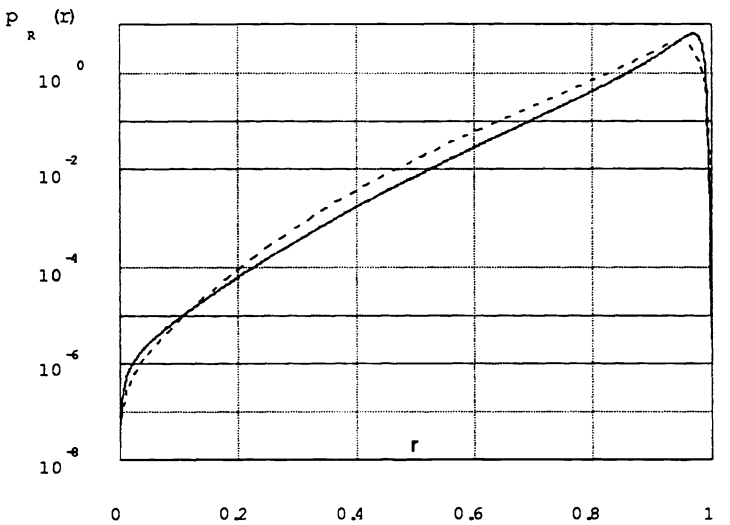

Figure 6: Marginal amplitude probability distribution at the output of a finite time integrator (full line) and a RC filter (dashed line). (Conditions : $D\tau_1^{FI} = 1$, $D\tau_1^{RC} = 3$, $a\tau_1^{RC} = 5.97$, grid 1000×500).

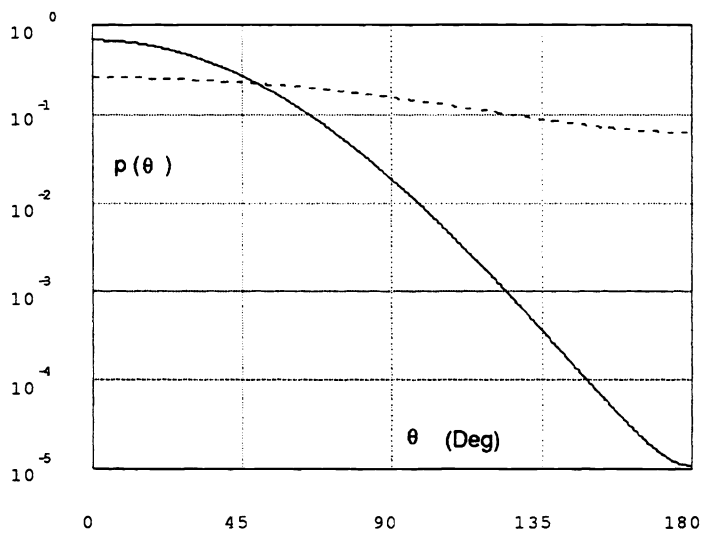

Figure 7: Marginal phase probability distribution at the output of a finite time integrator (full line) and a RC filter (dashed line). (Conditions : $D\tau_1^{FI} = 1$, $D\tau_1^{RC} = 3$, $a\tau_1^{RC} = 5.97$, grid 1000×500).