Mode Vector Modulation: Optimal Signal Sets With Geometric Shaping

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Abstract: For mode vector modulation, used in conjunction with direct-detection, we present geometrically-optimized signal sets that correspond to the densest sphere packing in the generalized Stokes space. We show that the best trade-off between spectral and energy efficiency occurs for simplex constellations.

1. Introduction

For link distances smaller than 10 km, optical transceivers using M-ary pulse amplitude modulation (M-PAM)/direct-detection are currently preferred due to their attractive price and low operating cost compared to their long-haul counterparts based on coherent detection [1]. The main disadvantage of M-PAM is that its energy consumption scales quadratically with M [2], since the M-PAM constellation is one-dimensional.

Stokes vector modulation (SVM) [3] allows for more energy-efficient signaling than M-PAM while it is still amenable to direct detection. This is achieved by spreading the constellation points in the three-dimensional Stokes space, as opposed to the one-dimensional M-PAM signal space.

To further increase energy efficiency, the use of a higher-dimensional Stokes space is beneficial. This can be achieved by extending the concept of SVM to jointly using N spatial degrees of freedom (SDOFs) over few-mode or multicore fibers or free space. We recently proposed such a generalization of SVM and we named it mode vector modulation (MVM) [4]. MVM consists in sending optical pulses over multiple spatial and polarization modes simultaneously with the same shape but different amplitudes and initial phases.

For SVM, several authors proposed signal sets that offer different trade-offs between spectral and energy efficiency. Early research focused on equipower signal sets (corresponding to the special case of Polarization Shift Keying (PolSK)). Geometric PolSK constellation shaping was performed numerically initially by Betti et al. [5] by maximizing the minimum Euclidean distance among signals in 3D Stokes space and then by Benedetto and Poggiolini [6] by using the exact symbol error probability of M-ary PolSK as an objective function. More recently, Morsy-Osman et al. [7] designed non-equipower SVM constellations based on the face-centered cubic (FCC) lattice to achieve maximum packing density, using the minimum Euclidean distance criterion.

In this paper, we present geometrically-optimized, equipower signal sets for MVM, used in conjunction with optically-preamplified direct-detection receivers. For simplicity, instead of minimizing the average symbol error probability, we solve the generalized Thomson problem, which consists in determining the configuration of M same charges constrained to the surface of a unit Poincaré hypersphere that minimize the electrostatic potential energy [8]. Then, to minimize the average bit error probability, we optimize the bit-to-symbol mapping to decrease the Hamming distance between closest neighbors as much as possible using simulated annealing [8]. We show that the best trade-off between spectral and energy efficiency for $N > 2$ occurs for simplex constellations.

2. Generalized Thomson problem

The equipower MVM signal set can be described by generalized unit Jones vectors $|s_m\rangle$, $m = 1, \ldots, M$, with N complex components representing the amplitudes and initial phases of the pulses launched over N SDOFs [9, 10]. Equivalently, the MVM signal set can be described by generalized unit Stokes vectors $\hat{s}_m$, $m = 1, \ldots, M$, with $N^2 - 1$ real components [9, 10]. Generalized unit Jones and Stokes vectors are linked by a quadratic transform that places some constraints on the Stokes components for $N > 2$ [9, 10].

Our goal here is to spread out the constellation points in the generalized Stokes space and thus improve the symbol error rate (SER) in MVM/direct-detection-based links. To facilitate numerical calculations, we adopt an objective function borrowed from electrostatics [11], where one assumes that the constellation points are identical charges on the surface of the Poincaré hypersphere (generalized Thomson problem). The gradient-descent method [12] is used for the minimization of the simplified objective function.

It is worth mentioning that there is an extensive literature on uniformly distributing M points on a sphere S^2 in $\mathbb{R}^3$ (see Saff and Kuijlaars [13] for a comprehensive survey). Global minima for the Thomson Problem for $N = 2$ and M up to approximately 1,000 can be found on the Cambridge website [14]. Closely related to Thomson's problem is the Tammes problem [13] whose goal is to find the arrangement of M points on a unit sphere which maximizes the minimum distance between any two points. Jasper et al. [15] studied the generalized Tammes problem in the complex projective space for small values of N, M.

3. Error probability asymptotics

For the qualitative interpretation of results in Sec. 4, we need to take a closer look at the evaluation of error probability. The asymptotic expression for the pairwise symbol error probability based on the union bound is [16]

$$\boxed{P_{e|s}^{m'|m} \sim \frac{1}{2}\frac{1}{\sqrt{\pi}}\sqrt{\frac{1+\gamma}{1-\gamma}}\frac{1}{\sqrt{\gamma}\sqrt{\gamma_s}}\exp\left[-\frac{\gamma_s(1-\gamma)}{2}\right].} \tag{1}$$

where γ_s is the symbol signal-to-noise ratio (SNR) per SDOF and $\gamma := |\langle s_m|s_{m'}\rangle|$.

For $M > N$, the Welch–Rankin bound on γ is written as [15] $\gamma \geq \sqrt{M-N/N(M-1)}$. The Welch–Rankin bound on γ is not tight when the signal set cardinality tends to infinity. Below, we estimate γ from geometric arguments.

Fig. 1(a) shows the optimal Thomson constellation and the partitioning of the sphere into Dirichlet (Voronoi) cells for $N = 2, M = 256$. In general, for $N = 2$ and for large M's, the Dirichlet cells for an optimal configuration are mostly hexagonal [13]. For simplicity, let's assume that the constellation points form an ideal hexagonal lattice. The unit cell in a two-dimensional hexagonal lattice is a regular hexagon of side d, where d is the minimum Euclidean distance between pairs of points. It comprises three net points, one at the center and six points at the vertices of the hexagon shared among three adjacent tiles each. The area of each unit cell is $\delta A = 3\sqrt{3}d^2/2$. We can estimate d if we divide the area of the unit sphere S^2, equal to $A = 4\pi$, by the total area of $M/3$ unit cells. We obtain the estimate $d^2 = 8\pi/(\sqrt{3}M)$. We observe that, in the asymptotic limit of large M's, the Euclidean distance is inversely proportional to the square root of the number of points M. By combining the formulas [17] $d^2 = \|\hat{s}-\hat{s}'\|^2 = 2(1-\hat{s}\cdot\hat{s}'), \gamma^2 = |\langle s|s'\rangle|^2 = (1+\hat{s}\cdot\hat{s}')/2$, and using the first-order Taylor expansion of the square root of γ^2 we obtain for the average symbol error probability $\bar{P}_{e|s} \sim \exp\left[-\pi\gamma_s/(2\sqrt{3}M)\right]$.

For Gray-like bit-to-symbol mapping, the average bit error probability is related to the average symbol error probability by $\bar{P}_{e|b} \simeq \bar{P}_{e|s}/k$, where $k := \log_2 M$, while in the case of quasi-orthogonal signal sets $\bar{P}_{e|b} \simeq M\bar{P}_{e|s}/[2(M-1)] \simeq \bar{P}_{e|s}/2$ [2]. We expect that Gray-like bit-to-symbol mapping will be applicable at large constellation cardinalities M, while orthogonal signal sets will occur for $M < N$ and quasi-orthogonal signal sets will occur for $N < M < N^2$. Asymptotically, the difference in SNR in dB for all these bit-to-symbol mappings is small and, for a qualitative interpretation of the results of Fig. 1(b), we can assume that $\bar{P}_{e|b} \simeq \bar{P}_{e|s}$.

Defining the MVM spectral efficiency per SDOF as $\eta := k/N$ and relating the symbol SNR per SDOF to the bit SNR per SDOF as $\gamma_s := k\gamma_b$ we can write, for a given average bit error probability, for $N = 2$ (SVM case)

$$\eta \sim \frac{\gamma_b(dB)}{20\log 2}. \tag{2}$$

Using a similar geometric argument for the case of $N > 2$ (MVM case), we find that $d^2 \sim M^{-\frac{1}{N-1}}$ and we can write, for a given average bit error probability,

$$\eta \sim \frac{N-1}{N}\frac{\gamma_b(dB)}{10\log 2}. \tag{3}$$

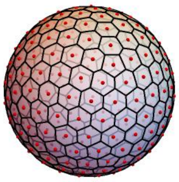
In other words, we expect that the slope $\eta/\gamma_b(dB)$ at large constellation cardinalities M will be 0.16 for $N = 2$ and will increase to 0.33 for $N \to \infty$, which is approximately the slope of the Shannon capacity formula.

At the other extreme, for orthogonal signal sets $M < N$, $\gamma = 0$ and we expect that, for a given $\bar{P}_{e|b}$,

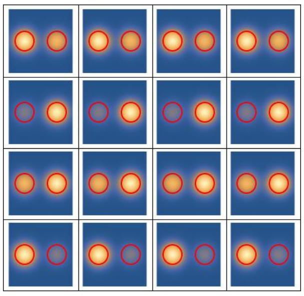
$$\eta \sim 10^{-\frac{\gamma_b(dB)}{10}}. \tag{4}$$

4. Results and discussion

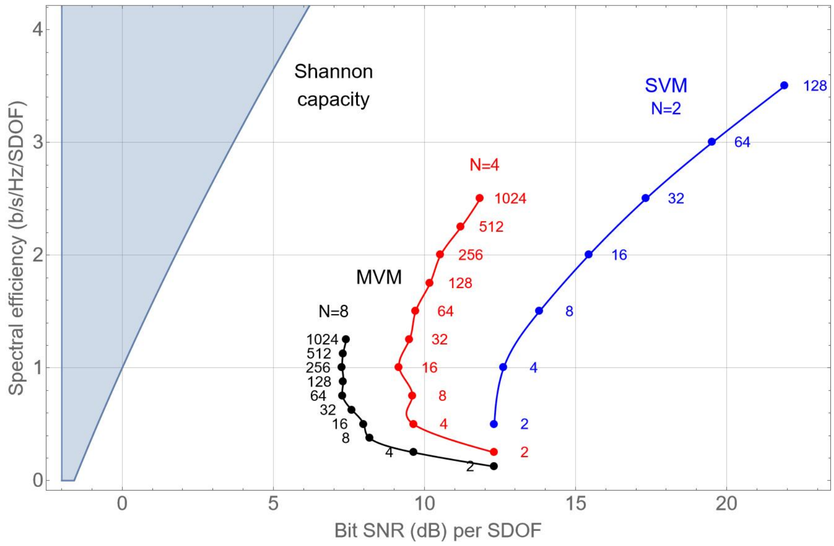
Fig. 1(b) shows the MVM spectral efficiency per SDOF vs the bit SNR per SDOF required to achieve a bit error probability of 10^{-4}. Each curve corresponds to a different number of degrees of freedom N, and each point within a curve corresponds to a different constellation cardinality M. It is worth stressing that these graphs represent geometrically-shaped constellations with optimized bit-to-symbol mapping. Suboptimal MVM constellations lie on the right of these graphs.



(a)



(c)



(b)

Fig. 1: (a) Optimized constellation and spherical Voronoi cells for $N = 2, M = 256$, obtained by solving the Thomson problem; (b) MVM spectral efficiencies per SDOF vs the bit SNR per SDOF at an average bit error probability of 10^{-4} for different degrees of freedom N and constellation cardinalities M; (c) Intensity plots of the optimal MVM signal set for $N = 4$, $M = 16$, over a two-core multicore fiber with identical uncoupled single-mode cores.

We can interpret the previous numerical results using the asymptotic analysis of Sec. 3. In general, we expect that the MVM spectral efficiency η as a function of the bit SNR per SDOF in dB $\gamma_b(dB)$ will be a C-shaped curve, with the upper section increasing linearly with $\gamma_b(dB)$ according to (2), (3), and the lower section decreasing exponentially with $\gamma_b(dB)$ according to (4). The apex of each C-shaped curve occurs for simplex constellations $M = N^2$, where $\gamma^2 = (N+1)^{-1}$. We conclude that simplex constellations represent the best compromise between spectral and energy efficiency for $N > 2$. A simplex signal set for $N = 4$, $M = 16$, is shown in Fig. 1(c).

5. Summary

In this paper, we presented geometrically-optimized M-ary MVM constellations transmitted over N SDOFs and studied their back-to-back performance in the amplified spontaneous emission (ASE) noise-limited regime. We showed that the best trade-off between spectral and energy efficiency for $N > 2$ occurs for simplex constellations.

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