Modeling of modal dispersion in multimode and multicore optical fibers

(Invited Paper)

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***Abstract*—Modal dispersion in strongly-coupled multimode and multicore optical fibers can be viewed as a generalization of polarization-mode dispersion in single-mode fibers. Due to the similarities between these two transmission effects, the conventional Jones and Stokes calculus for polarization-mode dispersion can be extended to the case of modal dispersion. In this paper, we review and expand the theoretical framework used for the representation of modal dispersion in Stokes space by the modal dispersion vector. We show, for the first time, that the modal dispersion vector can be written as a weighted sum of the Stokes vectors representing the principal modes with the corresponding mode group delays as coefficients. This constitutes a fundamental relationship that leads to a reinterpretation of the modal dispersion vector and can be used to derive its properties.**

Keywords – Optical fiber communication, multimode optical fiber, multicore optical fiber, modal dispersion, spatial division multiplexing.

I. Introduction

The global data traffic crossing the Internet is expected to increase with a compound annual growth rate (CAGR) exceeding 20% until 2020 [1], [2]. If this trend continues unabated for a longer period of time, it can lead to capacity exhaustion of the existing fiber-optic backbone network [3]. To accommodate future traffic demands, it is desirable to increase spectral efficiency in congested links by using spatial division multiplexing (SDM) to send independent data streams over disjoint spatial paths. This can be done, for instance, by using different modes of multimode optical fibers (MMFs) or different cores of multicore optical fibers (MCFs) [4], [5].

Among other transmission impairments, MMFs and MCFs exhibit modal dispersion (MD), mode-dependent loss (MDL), multipath interference, and intermodal/intercore crosstalk [6]. This paper is devoted to the accurate modeling of MD in strongly-coupled MMFs/MCFs in the absence of other modal effects. In this regime, MD can be viewed as a generalization of polarization-mode dispersion (PMD) in single-mode fibers (SMFs) [6]. Due to the similarities between MD and PMD, the PMD formalism expressed in the conventional Jones and Stokes spaces [7]-[11] was generalized into higher dimensions by several authors, e.g., [12]-[16]. MD in MMFs and MCFs can be fully described in the generalized Jones space by a set of orthogonal propagation modes called principal modes (PMs) and by their corresponding mode group delays (MGDs) [6]. Alternatively, MD can be represented geometrically by a vector in the generalized Stokes space, called the MD vector [12]. To the best of our knowledge, so far there has existed no explicit analytical relationship between the MD vector, the PMs, and the MGDs. In this paper, for the first time, we show that the MD vector can be expressed as a weighted sum of the Stokes vectors representing the PMs with the corresponding MGDs as coefficients. This leads to a new interpretation of how the MD vector encapsulates both the PMs and the MGDs in a single mathematical entity.

The rest of the paper is organized as follows: In Section II, we review the generalized Jones and Stokes formalism [12]-[16] for the modeling of MD in MMFs and MCFs. More specifically, we take a brief look in the generalized Jones vectors and matrices, the expansion of the latter in the basis of the generalized Gell-Mann matrices, the transition between the generalized Jones and Stokes spaces using the above expansion, and the properties of the vector dot products in both spaces. In Section III, we derive a concise analytical relationship that links the MD vector with the input PMs and the corresponding MGDs. In the Appendix, based on this new relationship, we rederive previously proposed analytical expressions for the norm of the MD vector and the projections of the MD vector on the PMs.

II. Theoretical background

A. Literature survey and motivation

Modeling of MD in long-haul SDM optical communications systems using MMFs/MCFs started circa 2005, when Fan and Kahn [17] first showed that the concept of principal states of polarization (PSPs) in SMFs [7] can be generalized to MMFs/MCFs. Later on, Ho and Kahn [6] used a generalized Jones matrix concatenation model to derive analytical expressions for the probability density functions of the MDGs and the MDL in strongly-coupled MMFs/MCFs. In 2012, Antonelli et al. [12] extended Gordon and Kogelnik's spinor PMD formalism [10] to the modeling of MD in strongly-coupled MMFs/MCFs. Several follow-up papers, e.g., [13]-[16], elucidated various facets of this formalism.

The aim of this invited paper is twofold: (i) To review the MD formalism and reconcile the differences in the mathematical conventions adopted by various authors [12]-[16]; (ii) To

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derive a new analytical relationship linking the MD vector to the PMs and their corresponding MGDs.

The MD vector is a generalization of the PMD vector for $N > 2$, where N is the number of spatial and polarization modes. Recall the conventional definition of the input PMD vector $\vec{\tau}_s$ as the product of the slow input PSP vector $\hat{p}$ in Stokes space with the differential group delay (DGD) τ between the two PSPs (Fig. 1) [10].

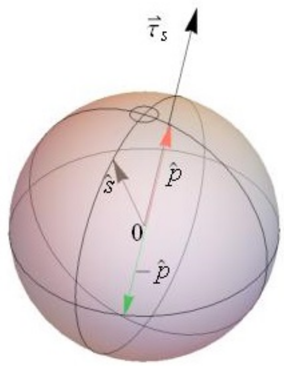

Fig. 1. Poincaré sphere, input slow and fast PSPs, $\hat{p}$ and $-\hat{p}$, respectively, and input PMD vector $\vec{\tau}_s$ in the case of SMF ($N = 2$). An arbitrary launch state of polarization (SOP) is denoted by $\hat{s}$.

Alternatively, it is possible to redefine the input PMD vector $\vec{\tau}_s$ as the sum of the Stokes vectors representing the slow and fast PSPs, $\hat{p}$ and $-\hat{p}$, respectively, with the corresponding group delays $\tau/2$, $-\tau/2$ as coefficients. This new definition yields a PMD vector identical to the conventional one

$$\vec{\tau}_s \triangleq \frac{\tau}{2}\hat{p} + \left(-\frac{\tau}{2}\right)(-\hat{p}) = \tau\hat{p}. \tag{1}$$

The advantage of this new definition is that it can be generalized in the case of higher dimensions $N > 2$, whereas the conventional definition of the PMD vector fails to scale with the number of modes.

More specifically, in Section III, we will show that, in all cases ($N \geq 2$), the input MD vector $\vec{\tau}_s$ can be written as a weighted sum of the Stokes vectors $\hat{p}_i$, representing the input PMs, with the corresponding MGDs τ_i, $i = 1, \ldots, N$ as coefficients [see expression (21) below].

B. Generalized Jones and Stokes spaces

The electric field of a monochromatic optical wave at a given time instant and position in a N-mode waveguide can be expressed as the vector sum

$$\mathbf{E}(\mathbf{r}, t) = \sum_{k=1}^{N} c_k \mathbf{E}_k(\mathbf{r}, t), \tag{2}$$

where $\mathbf{E}_k(\mathbf{r}, t)$ represent the electric fields of individual modes and the complex coefficients $c_k, k = 1, \ldots, N$ represent the mode excitations [12]. The latter satisfy the relationship

$$\sum_{k=1}^{N} |c_k|^2 = 1. \tag{3}$$

In the following, we use the methodology and the notation of [10], [11]: Dirac's bra-ket vectors represent vectors in the generalized Jones space and hats represent vectors in the generalized Stokes space.

We define the generalized unit Jones vectors as $|s\rangle \triangleq [c_1, \ldots, c_N]^T$, where T denotes the transpose of a matrix. Combinations of propagation modes are described by such vectors.

Linear optical devices are represented by $N \times N$ complex matrices called generalized Jones matrices, similar to the two-dimensional case. Their action results in a simple multiplication of the input Jones vector by the corresponding Jones matrix.

There are several advantages in using the generalized Stokes space instead of the generalized Jones space: The Stokes formalism allows to depict a combination of modes in a $(N^2 - 1)$-dimensional Stokes space as a point on the surface of a (N^2-2)-hypersphere (i.e., a generalized Poincaré sphere) with unit radius. This geometric representation is not as eloquent as in the case of the three-dimensional Stokes space, nevertheless, it has some aesthetic appeal. The Stokes space can also be used for the representation of MD in a concise form in terms of the MD vector. Moreover, a generalized Stokes vector can be directly measured using multiple mode filters and a power meter, similarly to the case $N = 2$ [18]. Finally, the MGD of narrowband optical pulses during propagation in MMFs/MCFs can be written simply as the inner product of the input MD vector with the Stokes vector corresponding to the launched combination of modes [16]. This relationship can be used to characterize MD using the mode-dependent signal delay method [16].

To transition to Stokes space, we will make use of the fact that any matrix in the N-dimensional Jones space can be decomposed in the basis of the $N \times N$ identity matrix $\mathbf{I}$ and the $N^2 - 1$ generalized Gell-Mann matrices $\lambda_1, \ldots, \lambda_{N^2-1}$ with dimensions $N \times N$ [19], [20]. The latter can be constructed as follows: Consider an arbitrary orthonormal basis in Jones space $|b_1\rangle, \ldots, |b_N\rangle$. We first define the following auxiliary symmetric, antisymmetric, and diagonal matrices, respectively [19], [20]:

$$\mathbf{U}_{jk} \triangleq |b_j\rangle\langle b_k| + |b_k\rangle\langle b_j|, \tag{4}$$

$$\mathbf{V}_{jk} \triangleq -i\left(|b_j\rangle\langle b_k| - |b_k\rangle\langle b_j|\right), \tag{5}$$

$$\mathbf{W}_l \triangleq -\sqrt{\frac{2}{l(l+1)}}\left(l\,|b_{l+1}\rangle\langle b_{l+1}| - \sum_{j=1}^{l}|b_j\rangle\langle b_j|\right), \tag{6}$$

for given indices j, k, l.

Then, we define the sets [19], [20]

$$\begin{aligned} \mathcal{U} &\triangleq \{\mathbf{U}_{jk} : 1 \leq j < k \leq N\}, \\ \mathcal{V} &\triangleq \{\mathbf{V}_{jk} : 1 \leq j < k \leq N\}, \\ \mathcal{W} &\triangleq \{\mathbf{W}_\ell : 1 \leq l \leq N - 1\}. \end{aligned} \tag{7}$$

The generalized Gell-Mann matrices λ_i are the elements of the union of the above sets, i.e., $\lambda_i \in \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$, $i =$

$1, \ldots, N^2-1$ [19], [20]. The order in which the elements λ_i are listed is immaterial since reordering them results in a permutation of the Stokes vector components in (12).

From their definition, we note that the generalized Gell-Mann matrices are traceless and mutually trace-orthogonal [19], [20]

$$\begin{aligned} &\operatorname{Tr}(\lambda_i) = 0, \\ &\operatorname{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}, \end{aligned} \tag{8}$$

where $\operatorname{Tr}(.)$ denotes the trace operator and $\delta_{ij},\ i,j = 1, \ldots, N^2-1$, denotes the Kronecker delta.

To write concisely the expansion of a Jones matrix in the orthonormal basis of the identity matrix and the N^2-1 generalized Gell-Mann matrices, we also define the Gell-Mann spin vector $\mathbf{\Lambda} \triangleq [\lambda_1, \ldots, \lambda_{N^2-1}]^T$, in analogy to the Pauli spin vector [10].

To illustrate the expansion of a Jones matrix in the orthonormal basis of the identity matrix and the N^2-1 generalized Gell-Mann matrices, following the methodology of [10], we first define the dyadic operator $|p\rangle\langle q|$ as the outer product of two generalized Jones vectors

$$\mathbf{\Xi} \triangleq |p\rangle\langle q| = \begin{bmatrix} p_1{q_1}^* & p_1{q_2}^* & \cdots & p_1{q_N}^* \\ p_2{q_1}^* & p_2{q_2}^* & \cdots & p_2{q_N}^* \\ \vdots & \vdots & \ddots & \vdots \\ p_N{q_1}^* & p_N{q_2}^* & \cdots & p_N{q_N}^* \end{bmatrix}, \tag{9}$$

where the asterisk denotes complex conjugate.

A special case of a dyadic operator is the projection operator. This represents a mode filter, i.e., the equivalent of a polarizer in the two-dimensional case

$$\rho \triangleq |s\rangle\langle s| = \begin{bmatrix} |c_1|^2 & c_1{c_2}^* & \cdots & c_1{c_N}^* \\ c_2{c_1}^* & |c_2|^2 & \cdots & c_2{c_N}^* \\ \vdots & \vdots & \ddots & \vdots \\ c_N{c_1}^* & c_N{c_2}^* & \cdots & |c_N|^2 \end{bmatrix}. \tag{10}$$

Using (4)-(8), we expand the projection operator in the orthonormal basis of the identity matrix and the N^2-1 generalized Gell-Mann matrices $\{\mathbf{I}, \lambda_1, \ldots, \lambda_{N^2-1}\}$ [13], [15]

$$|s\rangle\langle s| = \frac{1}{N}\left[\mathbf{I} + \sqrt{\frac{N(N-1)}{2}}\,\hat{s}\mathbf{\Lambda}\right], \tag{11}$$

where we defined the generalized Stokes vectors as [13], [15]

$$\hat{s} \triangleq \sqrt{\frac{N}{2(N-1)}}\,\langle s|\,\mathbf{\Lambda}\,|s\rangle\,. \tag{12}$$

The normalization coefficient $\sqrt{N/[2(N-1)]}$ in (12) is chosen such that $\|\hat{s}\| = 1$.

It should be emphasized that Antonelli et al. [12] used different multiplication coefficients for (11), (12). Here, we adopt the conventions of [13]-[15] and [19] due to their backward compatibility with the PMD case [10], [11].

One can use the following eigenvalue equation as an inverse transform from Stokes to Jones space [11]

$$\sqrt{\frac{N}{2(N-1)}}\,(\hat{s}\mathbf{\Lambda})\,|s\rangle = |s\rangle\,. \tag{13}$$

Equation (13) stems from the expansion of the projection operator (10) on the basis of the identity matrix and the N^2-1 Gell-Mann matrices (11) by multiplying the latter with $|s\rangle$ from the right and rearranging the terms. It indicates that the Jones vector $|s\rangle$ corresponding to the Stokes vector $\hat{s}$ is the eigenvector of the operator $(\hat{s}\mathbf{\Lambda})$ corresponding to the $\sqrt{2(N-1)/N}$ eigenvalue. Most points on the generalized Poincaré sphere do not satisfy (13), which means that they do not correspond to valid combinations of modes.

The connection between the dot products in Jones and Stokes space is the following [19]:

$$|\langle q \mid p\rangle|^2 = \frac{1}{N}\left[1 + (N-1)\,\hat{p}\hat{q}\right]. \tag{14}$$

To obtain (14), we first write the dyadic operator (9) as a linear combination of the identity matrix and the N^2-1 Gell-Mann matrices and then we multiply with $|p\rangle$ from the right and $\langle q|$ from the left.

Orthogonal vectors in Jones space correspond to non-orthogonal vectors in Stokes space. Setting $\langle q \mid p\rangle = 0$ in (14), we obtain [19]

$$\hat{p}\hat{q} = -\frac{1}{N-1}. \tag{15}$$

The dot product property (15) is satisfied by the N vectors connecting the origin of the axes with the N vertices of a $(N-1)$-dimensional regular simplex centered at the origin and inscribed in the unit sphere S^{N-2} [19]. Conversely, this indicates that the N vectors of an orthonormal basis in Jones space are mapped into Stokes vectors that form the vertices of such $(N-1)$-dimensional regular simplex [19] (cf. Fig. 3).

C. Modeling long MMFs/MCFs

A long linear MMF/MCF link can be modeled as a concatenation of independent short fiber segments [6]. The latter can be implemented using random unitary matrices for the coupling sections and diagonal unitary matrices for the delay sections (Fig. 2).

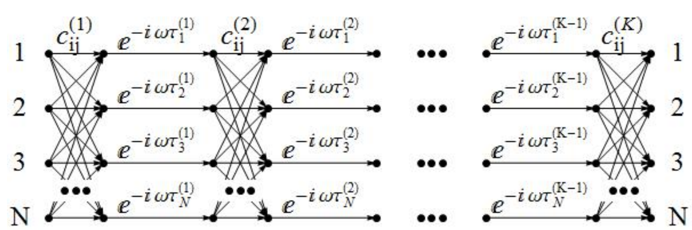

Fig. 2. Trellis diagram visualization of the simulation model of a long MMF/MCF link with N modes/cores composed of a concatenation of K-1 uncoupled short uniform fiber segments with K coupling stages in between. All functionalities can be implemented using $N \times N$ unitary matrices in Jones space [6]. (Symbols: Nodes = fiber modes, $\tau_i^{(k-1)}$ = MGD of the $i-$th mode in the $k-$th segment, $c_{i,j}^{(k)}$ = coupling coefficient between the $i-$th and the $j-th$ modes in the $k-$th segment).

From the directed graph shown in Fig. 2, we can write the following matrix equation that connects the input $|s\rangle$ and output $|t(\omega)\rangle$ Jones vectors [10]

$$|t(\omega)\rangle = \mathbf{U}(\omega)\,|s\rangle\,, \tag{16}$$

where $\mathbf{U}(\omega)$ is the unitary transfer matrix of the fiber in Jones space.

The input MD vector $\vec{\tau}_s$ is defined as

$$i\mathbf{U}(\omega)^\dagger \mathbf{U}_\omega(\omega) \triangleq \sqrt{\frac{N-1}{2N}}\vec{\tau}_s(\omega)\mathbf{\Lambda}, \tag{17}$$

where the index ω denotes differentiation with respect to the angular frequency and $\dagger$ denotes the adjoint matrix.

It is worth pointing out that Antonelli et al. [12] and Milione et al. [16] use different multiplication coefficients for the RHS of (17). Our choice stems from (11), (12) and is backwards compatible with the PMD case [10], [11].

The input PMs are the eigenstates of the operator [10]

$$i\mathbf{U}(\omega)^\dagger \mathbf{U}_\omega(\omega) \left|p_i(\omega)\right\rangle \triangleq \tau_i(\omega) \left|p_i(\omega)\right\rangle, \tag{18}$$

where $\tau_i(\omega)$, $i = 1, \ldots, N$, are the corresponding MGDs. The output PMs are given by $\left|q_i(\omega)\right\rangle \triangleq \mathbf{U}(\omega)\left|p_i(\omega)\right\rangle$, $i = 1, \ldots, N$.

Millione et al. [16] showed that the group delay τ_g of an optical pulse corresponding to a given combination of launch modes is related to the dot product of the input MD vector $\vec{\tau}_s(\omega)$ and the input launch state $\hat{s}$ in Stokes space. We rewrite their expression in a slightly modified form:

$$\tau_g = \frac{N-1}{N}\left\langle \vec{\tau}_s(\omega)\right\rangle \hat{s}, \tag{19}$$

where angle brackets denote spectral averaging [10].

It is worth noting that (19) differs from the original expression (16) of Millione et al. [16] on several points: there is a corrective multiplicative factor of $(N-1)/N$, the input MD vector $\vec{\tau}_s$ is spectrally-averaged, and the average MGD is assumed zero for convenience [12]

$$\sum_{i=1}^{N} \tau_i(\omega) = 0. \tag{20}$$

The above changes are necessary in order to make (19) compatible with (11), (12) and with expression [5.30] in [10].

III. Modeling of modal dispersion

We will show that, in the absence of MDL, the input MD vector can be written as a weighted sum of the Stokes vectors representing the input PMs with the corresponding MGDs as coefficients

$$\boxed{\vec{\tau}_s(\omega) = \sum_{i=1}^{N} \tau_i(\omega)\,\hat{p}_i(\omega)\,.} \tag{21}$$

Since the input and output PMs form orthonormal bases in Jones space, according to (15), they are mapped into Stokes vectors that form the vertices of $(N-1)$-dimensional regular simplices. It is possible to visualize these simplices in the case of bimodal fibers ($N = 2$) (i.e., the PSPs $\hat{p}$, $-\hat{p}$ form a straight line in Fig. 1), as well as in the case of hypothetical trimodal ($N = 3$) and quadrimodal ($N = 4$) fibers (Fig. 3), where the PMs form an equilateral triangle and a regular tetrahedron, respectively. In general, the Stokes vector in the direction of the MD vector does not coincide with a state that corresponds to a valid combination of modes in Jones space, i.e., the eigenvalue equation (13) is not satisfied.

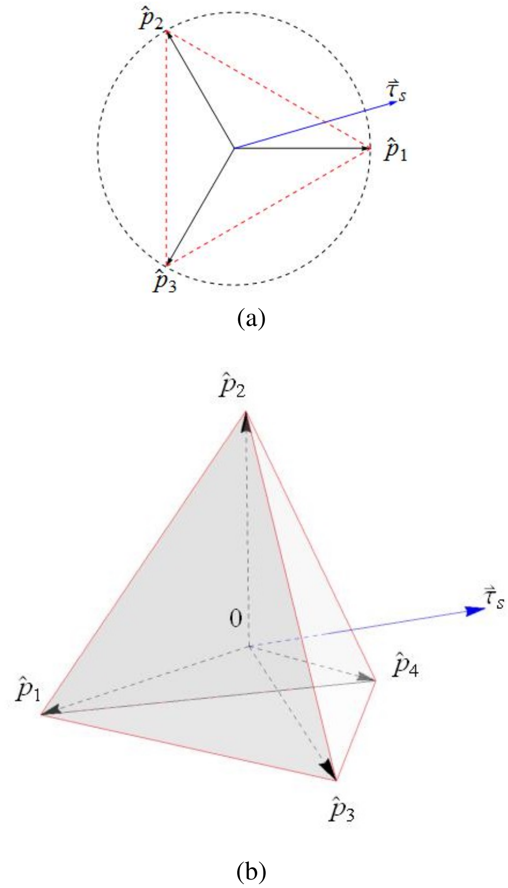

Fig. 3. (a) For a hypothetical trimodal fiber, the Stokes space is eight dimensional. The input PMs in Stokes space are vectors from the origin of the axes to the vertices of an equilateral triangle. The input MD vector is on the same plane as the input PMs. The dotted circle indicates the boundaries of the generalized Poincaré sphere; (b) For a hypothetical quadrimodal fiber, the Stokes space is 15-dimensional. The input PMs in Stokes space are vectors from the origin of the axes to the vertices of a regular tetrahedron. The input MD vector lies in the same 3D subspace as the input PMs.

To prove (21), we will first derive an alternative expression for (19). For this purpose, we introduce the Jones transfer matrix decomposition

$$\mathbf{U}(\omega) = \mathbf{Q}(\omega)\,\mathbf{D}(\omega)\,\mathbf{P}^\dagger(\omega)\,, \tag{22}$$

where $\mathbf{D}(\omega)$ is a diagonal matrix containing the mode group delays of the PMs

$$\mathbf{D}(\omega) = \begin{bmatrix} e^{-i\omega\tau_1(\omega)} & & 0 \\ & \ddots & \\ 0 & & e^{-i\omega\tau_N(\omega)} \end{bmatrix}, \tag{23}$$

and $\mathbf{P}(\omega)\,, \mathbf{Q}(\omega)$ are unitary matrices with columns the input and output PMs, respectively.

Consider a scalar pulse $g(t)$ with unit energy, which is originally centered on $t = 0$. These conditions can be mathematically expressed as follows:

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = 1,$$
$$\int_{-\infty}^{\infty} t|g(t)|^2 dt = 0.$$

We launch this pulse into a MMF using a combination of modes $|s\rangle$. The output electric field spectrum can be written in equivalent baseband representation as

$$\mathbf{E}(\omega) = G(\omega)\,\mathbf{U}(\omega)\,|s\rangle\,. \tag{24}$$

Substituting expression (22) for the fiber transfer matrix into (24) yields

$$\mathbf{U}(\omega) = \sum_{k=1}^{N} e^{-i\omega\tau_k(\omega)}\,|q_k(\omega)\rangle\,\langle p_k(\omega)|. \tag{25}$$

Assuming that the pulse has sufficiently narrow spectrum around $\omega = 0$ such that all frequency-dependent terms in the RHS of (25) are constant, the output electric field in the time domain can be written as

$$\mathbf{E}(t) = \sum_{k=1}^{N} g(t-\tau_k)\,\langle p_k \mid s\rangle\,|q_k\rangle\,, \tag{26}$$

where we omitted the dependence from $\omega = 0$ to avoid clutter.

The pulse group delay τ_g is defined as [10]

$$\tau_g \triangleq \int_{-\infty}^{\infty} t\mathbf{E}(t)^{\dagger}\mathbf{E}(t)\,dt, \tag{27}$$

which yields

$$\tau_g = \sum_{i=1}^{N} \tau_i |\langle p_k \mid s\rangle|^2 = \frac{1}{N}\sum_{i=1}^{N}\tau_i + \frac{N-1}{N}\sum_{i=1}^{N}\tau_i \hat{p}_i \hat{s}. \tag{28}$$

Substituting (20) into (28), we obtain

$$\tau_g = \frac{N-1}{N}\sum_{i=1}^{N}\tau_i \hat{p}_i \hat{s}. \tag{29}$$

Finally, by comparison of (19) and (29), for a sufficiently narrowband pulse such that $\langle \vec{\tau}_s(0)\rangle = \vec{\tau}_s(0) = \vec{\tau}_s$, we arrive at the desired expression

$$\vec{\tau}_s = \sum_{i=1}^{N}\tau_i \hat{p}_i. \tag{30}$$

Q.E.D.

IV. Summary

In this paper, we reviewed and expanded the theoretical framework used for the representation of MD in Stokes space by the MD vector. We reconciled the differences in the mathematical conventions adopted by various authors [12]-[16]. Finally, we showed that, in the absence of MDL, the input MD vector can be written as a weighted sum of the Stokes vectors representing the input PMs with the corresponding MGDs as coefficients.

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Appendix
MD vector properties

In this Appendix, we will use our definition of the input MD vector (21) to prove the MD vector properties given by the relationships (31) and (32) below. These expressions were initially derived by Antonelli et al. [12] using a different method. Here, they differ slightly from their original form in [12] due to our choice of multiplication coefficients in (11), (12), and (17).

More specifically, we will show that:

(i) The norm of the input MD vector $\vec{\tau}_s(\omega)$ is given by

$$\|\vec{\tau}_s(\omega)\| = \sqrt{\frac{N}{N-1}\sum_{k=1}^{N}\tau_k^2(\omega)}. \tag{31}$$

(ii) The input MD vector $\vec{\tau}_s(\omega)$ is not aligned to any particular input PM $\hat{p}_{s,i}$. Its projection on the input PMs is given by

$$\vec{\tau}_s(\omega)\hat{p}_{s,i}(\omega) = \frac{N}{N-1}\tau_i(\omega) \tag{32}$$

where $i = 1, \ldots, N$.

We will first prove relationship (31). From (21), the squared magnitude of the MD vector is given by

$$\begin{aligned}\|\vec{\tau}_s(\omega)\|^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\tau_i(\omega)\,\tau_j(\omega)\,\hat{p}_i(\omega)\hat{p}_j(\omega)\\ &= \sum_{i=1}^{N}\tau_i^2(\omega) + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N}\tau_i(\omega)\,\tau_j(\omega)\,\hat{p}_i(\omega)\hat{p}_j(\omega)\,.\end{aligned}$$

Using (15) we obtain

$$\|\vec{\tau}_s(\omega)\|^2 = \sum_{i=1}^{N}\tau_i^2(\omega) - \frac{2}{N-1}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\tau_i(\omega)\,\tau_j(\omega). \tag{33}$$

By squaring (20), we get

$$2\sum_{i=1}^{N}\sum_{j=i+1}^{N}\tau_i(\omega)\,\tau_j(\omega) = -\sum_{i=1}^{N}\tau_i^2(\omega). \tag{34}$$

Substituting (34) into (33), we obtain

$$\left\|\vec{\tau}_s(\omega)\right\|^2 = \frac{N}{N-1}\sum_{i=1}^{N}\tau_i^2(\omega).$$

and thus (31) is proved.

Q.E.D.

Next, we will prove relationship (32). First, we multiply both sides of (21) with $\hat{p}_i$

$$\vec{\tau}_s(\omega)\,\hat{p}_i(\omega) = \sum_{j=1}^{N}\tau_j(\omega)\,\hat{p}_j(\omega)\,\hat{p}_i(\omega). \tag{35}$$

Substituting (15) into (35) yields

$$\vec{\tau}_s(\omega)\,\hat{p}_i(\omega) = \tau_i(\omega) - \frac{1}{N-1}\sum_{j\neq i}^{N}\tau_j(\omega). \tag{36}$$

Rearranging the terms in (20) yields

$$\sum_{j\neq i}^{N}\tau_j(\omega) = -\tau_i(\omega). \tag{37}$$

Substituting (37) into (36), we obtain the desired expression

$$\vec{\tau}_s(\omega)\,\hat{p}_i(\omega) = \frac{N}{N-1}\tau_i(\omega).$$

Q.E.D.

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