Introduction to Parameter Index Matrices & Design Matrices

In Program MARK, the Parameter Index Matrices (PIM) establish the set of real parameters that will be estimated. The numbering of the PIMs can also be used to place constraints on the real parameter estimates. When you start a new data analysis in Program MARK, you indicate what type of data you'll be working with and how many groups you'll be entering data for. Entering data by group is one (but not the only) way of entering categorical covariate information into MARK.

You will have one parameter index matrix for each parameter type for each group, and Program MARK has a graphical window for each parameter matrix that exists for a given problem. For known-fate data and 3 groups of animals (e.g., control group, treatment 1 group, treatment 2 group), you will have 3 matrices: each matrix will be for the survival rates for one of the groups. For live recaptures and 2 groups, you will have 2 Apparent Survival (phi) matrices and 2 Recapture Probability (p) matrices. The default set-up of the numbers is for them to be numbered to represent a group x time model.

For our initial problem, we'll work with known-fate data for one group. Thus, you will have one matrix: true survival matrix for the one group. The matrix will be a single row with as many columns as you have time intervals for estimating survival. When setting up a new analysis problem, you will also tell Program MARK how many time intervals you have data for in your input file. Our first problem will have data for 8 week-long intervals.

1	2	3	4	5	6	7	8

This model allows you to run a model for which survival is allowed to be different each week. You could change all the numbers to be "1". If you did so, you would constrain all 8 weekly survival rates to be the same.

1	1	1	1	1	1	1	1

When you go to run a model in Program MARK, the values in the PIMs establish how many rows there are in the Design Matrix: there will be 1 row for each unique value. If there are 8 different values in the PIMs, the Design Matrix will have 8 rows; if there is only 1 value, there will be 1 row. The next step is to decide how many columns you want in the Design Matrix and what to put in each column. One simple Design Matrix that you'll use commonly is an identity matrix (1's on the main diagonal and 0's everywhere else). For a PIM with 8 different numbers

1	2	3	4	5	6	7	8

for a model of known-fate survival over 8 different time intervals, you could model the survival rate for each interval with an identity matrix like the one below. The Design Matrix would indicate that you would like to estimate 8 β's and that each would be multiplied by a 1 or a 0. Survival rate over the 1st interval would be estimated as $\hat{\beta}_1 \cdot 1$ (all other β's x 0 = 0).

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    1    0    0    0    0    0    0    0
[2,]    0    1    0    0    0    0    0    0
[3,]    0    0    1    0    0    0    0    0
[4,]    0    0    0    1    0    0    0    0
[5,]    0    0    0    0    1    0    0    0
[6,]    0    0    0    0    0    1    0    0
[7,]    0    0    0    0    0    0    1    0
[8,]    0    0    0    0    0    0    0    1
```

In the material that follows, we'll explore ideas of how you might set up Design Matrices in various ways to achieve models of interest.

Consider the following dataset = 3 groups & 3 intervals

known fate group=1;
55 33;
55 38;
55 31;
known fate group=2;
55 43;
55 25;
55 30;
known fate group=3;
55 33;
55 29;
55 26;

PIM = Parameter Index Matrix – for this problem, let's use the following PIM's for the 3 groups.

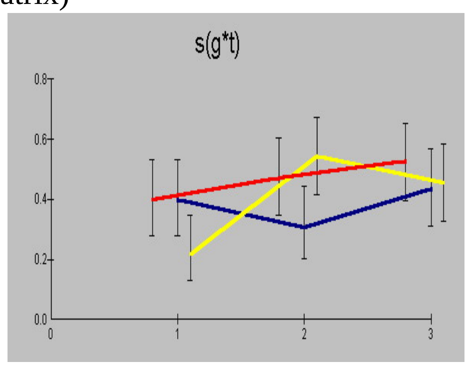
PIM – group 1: 1 2 3
PIM - group 2: 4 5 6
PIM - group 3: 7 8 9

This requests that 9 parameters be estimated, i.e., the maximum possible number given the dataset. We can then use the design matrix (a matrix of X values) to constrain the estimates in various ways. The design matrix is multiplied by the vector of beta's or coefficients being estimated for the given model. There are as many beta's estimated as there are non-zero columns, i.e., columns containing non-zero values.

S(g x t) – no additional constraints (identity matrix)

1. 1 0 0 0 0 0 0 0 0
2. 0 1 0 0 0 0 0 0 0
3. 0 0 1 0 0 0 0 0 0
4. 0 0 0 1 0 0 0 0 0
5. 0 0 0 0 1 0 0 0 0
6. 0 0 0 0 0 1 0 0 0
7. 0 0 0 0 0 0 1 0 0
8. 0 0 0 0 0 0 0 1 0
9. 0 0 0 0 0 0 0 0 1

$$\times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{pmatrix}$$



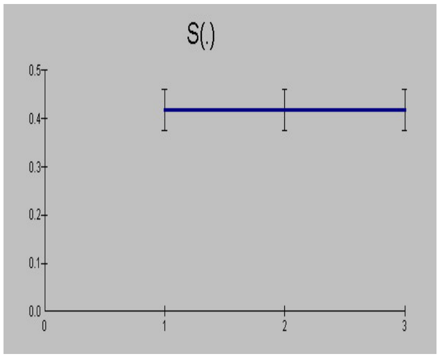
If a logit link were used, then we would obtain the following equations for estimating each of the 9 parameters of this model

1. group 1, week 1: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
2. group 1, week 2: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
3. group 1, week 3: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
4. group 2, week 1: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 1 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
5. group 2, week 2: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 1 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
6. group 2, week 3: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 1 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
7. group 3, week 1: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 1 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 0$
8. group 3, week 2: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 1 + \hat{\beta}_9 \cdot 0$
9. group 3, week 3: $\hat{\beta}_1 \cdot 0 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0 + \hat{\beta}_6 \cdot 0 + \hat{\beta}_7 \cdot 0 + \hat{\beta}_8 \cdot 0 + \hat{\beta}_9 \cdot 1$

S(.) – all 9 estimates are equal

1. 1
2. 1
3. 1
4. 1
5. 1
6. 1
7. 1
8. 1
9. 1

$\times \left(\beta_1 \right)$

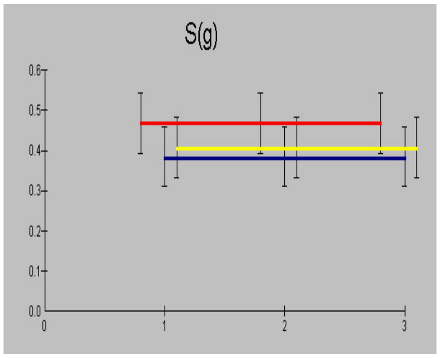


1-9 (all groups, all weeks): E[ln(S/(1-S)]= $\hat{\beta}_1 \cdot 1$

S(g) – estimates vary by group (constant across weeks)

1. 1 1 0
2. 1 1 0
3. 1 1 0
4. 1 0 1
5. 1 0 1
6. 1 0 1
7. 1 0 0
8. 1 0 0
9. 1 0 0

$\times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$



1, 2, & 3: group 1 (each week): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0$

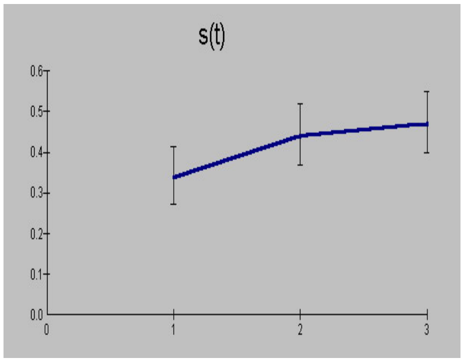
4, 5, & 6: group 2 (each week): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1$

7, 8, & 9: group 3 (each week): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0$

S(t) – estimates vary by week (constant across groups)

1. 1 1 0
2. 1 0 1
3. 1 0 0
4. 1 1 0
5. 1 0 1
6. 1 0 0
7. 1 1 0
8. 1 0 1
9. 1 0 0

$\times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$



1, 4, & 7: week 1 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0$

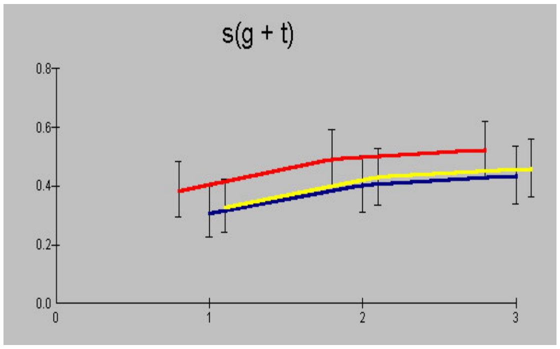
2, 5, & 8: week 2 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1$

3, 6, & 9: week 3 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0$

S(g + t) – estimates vary by time and group; in terms of the logit of S, there is a constant difference between the groups

1. 1 1 0 1 0
2. 1 1 0 0 1
3. 1 1 0 0 0
4. 1 0 1 1 0
5. 1 0 1 0 1
6. 1 0 1 0 0
7. 1 0 0 1 0
8. 1 0 0 0 1
9. 1 0 0 0 0

$$\times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}$$

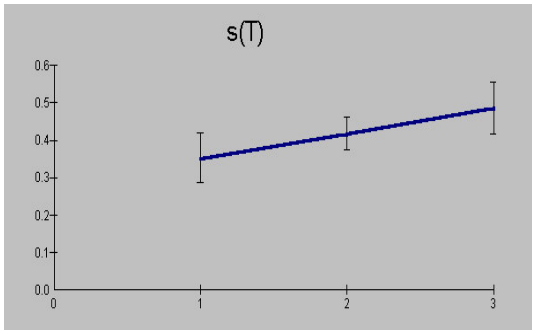


1. group 1, week 1: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 1 + \hat{\beta}_5 \cdot 0$
2. group 1, week 2: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 1$
3. group 1, week 3: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0$
4. group 2, week 1: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1 + \hat{\beta}_4 \cdot 1 + \hat{\beta}_5 \cdot 0$
5. group 2, week 2: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 1$
6. group 2, week 3: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0$
7. group 3, week 1: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 1 + \hat{\beta}_5 \cdot 0$
8. group 3, week 2: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 1$
9. group 3, week 3: $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0 + \hat{\beta}_4 \cdot 0 + \hat{\beta}_5 \cdot 0$

S(Trend) or S(T) – there is a linear TREND in the log-odds of survival

1. 1 1
2. 1 2
3. 1 3
4. 1 1
5. 1 2
6. 1 3
7. 1 1
8. 1 2
9. 1 3

$$\times \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$



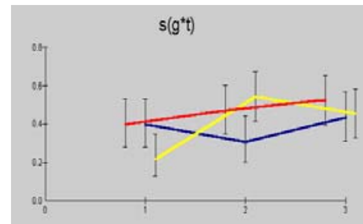
1, 4, & 7: week 1 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1$

2, 5, & 8: week 2 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 2$

3, 6, & 9: week 3 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 3$

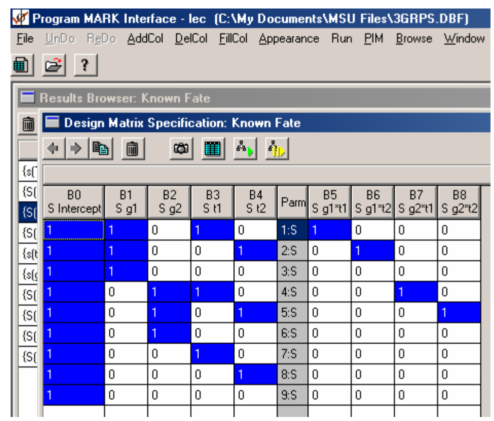
S(g x t) – built this time with g, t, and interaction terms (as opposed to the intercept-only or identity-matrix version we saw earlier). The design matrix includes 9 columns (X's) representing: (1) Intercept, (2) g1 (dummy variable indicating whether or not the observation is for group 1), (3) g2 (group 2 indicator), (4) t1 (time period 1 indicator), (5) t2 (time period 2 indicator), (6) g1*t1 (1st interaction term). (7) g1*t2 (2nd interaction term), (8) g2*t1 (3rd interaction term), and (9) g2*t2 (4th interaction term).

	Int	g1	g2	t1	t2	g1*t1	g1*t2	g2*t1	g2*t2
1.	1	1	0	1	0	1	0	0	0
2.	1	1	0	0	1	0	1	0	0
3.	1	1	0	0	0	0	0	0	0
4.	1	0	1	1	0	0	0	1	0
5.	1	0	1	0	1	0	0	0	1
6.	1	0	1	0	0	0	0	0	0
7.	1	0	0	1	0	0	0	0	0
8.	1	0	0	0	1	0	0	0	0
9.	1	0	0	0	0	0	0	0	0



When you're working in MARK, the design matrix will look like the one to the right.

MARK labels the columns in such a way that you can see which Beta is multiplied by each column. In the "Parm" column (the gray column in the middle), MARK also indicates which parameter (as numbered in the PIM goes with each row.



LOGIT Link Function Parameters of {S(g*t) Design Matrix}

Parameter	Beta	Standard Error	Lower 95% CI	Upper 95% CI
1:S Intercept	0.1091989	0.2700821	-0.4201619	0.6385598
2:S g1	-0.3651322	0.3832349	-1.1162726	0.3860082
3:S g2	-0.2915206	0.3824624	-1.0411470	0.4581058
4:S t1	-0.5146640	0.3856188	-1.2704769	0.2411489
5:S t2	-0.2183984	0.3819538	-0.9670279	0.5302310
6:S g1*t1	0.3651321	0.5462455	-0.7055090	1.4357733
7:S g1*t2	-0.3300411	0.5522249	-1.4124019	0.7523198
8:S g2*t1	-0.5793077	0.5732574	-1.7028923	0.5442769
9:S g2*t2	0.5830418	0.5408837	-0.4770902	1.6431739

This is a useful way to build the model because now you have estimated beta's that correspond to each of the terms in a more meaningful way than they do when you simply work with the 9 x 9 identity matrix. Of course, they give you identical estimates of the S_i.

Parameter	Estimate	Standard Error	LCI	UCI
1:S	0.4000000	0.0660578	0.2799041	0.5334517
2:S	0.3090909	0.0623121	0.2016081	0.4421425
3:S	0.4363637	0.0668717	0.3124179	0.5688040
4:S	0.2181818	0.0556905	0.1282866	0.3460624
5:S	0.5454546	0.0671408	0.4137586	0.6710841
6:S	0.4545454	0.0671408	0.3289160	0.5862413
7:S	0.4000000	0.0660578	0.2799041	0.5334518
8:S	0.4727272	0.0673196	0.3455720	0.6035219
9:S	0.5272726	0.0673196	0.3964780	0.6544278

For some of these models, you could simplify the PIMs so as to simplify the design matrix. For S(t) and S(T), if all 3 PIMS were set to 1 2 3; then you would have a simpler design matrix (3 x 3).

Let's use the following PIM's for the 3 groups.

PIM – group 1: 1 2 3
PIM - group 2: 1 2 3
PIM - group 3: 1 2 3

S(t) – S varies by week and is not constrained to follow a specified pattern

1. 1 0 0
2. 0 1 0
3. 0 0 1

$$\times \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

week 1 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1 + \hat{\beta}_3 \cdot 0$
week 2 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 1$
week 3 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 0 + \hat{\beta}_3 \cdot 0$

S(T) – S is constrained such that the log-odds of S follows a linear trend

1. 1 1
2. 1 2
3. 1 3

$$\times \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

week 1 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 1$
week 2 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 2$
week 3 (all groups): $\hat{\beta}_1 \cdot 1 + \hat{\beta}_2 \cdot 3$

Finally, realize that you could use the following PIM's for the 3 groups to collapse all the data.

PIM – group 1: 1 1 1
PIM - group 2: 1 1 1
PIM - group 3: 1 1 1

S(.)

1. 1 $\times (\beta_1)$

all groups, all weeks: E[ln(S/(1-S)]= $\hat{\beta}_1 \cdot 1$