Returning to the incremental section, we will now solve for V and I using circuit laws. We will assume time-harmonic excitation.

$$v(z,t) = v(z)\cos(\omega t)$$

$$v(z,t) = \mathrm{Re}\left\{V_s(z)e^{j\omega t}\right\}$$

$$\frac{\partial v(z,t)}{\partial t} = j\omega V_s(z)$$

Applying KVL:

$$-v(z,t) + i(z,t)R\Delta z + L\Delta z\frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t) = 0$$

Applying KCL:

$$i(z+\Delta z,t) - i(z,t) + v(z+\Delta z,t)G\Delta z + C\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t} = 0$$

Rearranging each equation, dividing by Δz, and taking the limit as Δz→0, we find…

$$\lim_{\Delta z\to 0}\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -i(z,t)R-L\frac{\partial i(z,t)}{\partial t}$$

$$\lim_{\Delta z\to 0}\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = -Gv(z+\Delta z,t)-C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Recognizing that the left hand side of the equations define a derivative, we have the

‘Telegrapher’ of ‘Transmission Line’ equations:

$$\boxed{\begin{aligned}\frac{\partial v(z,t)}{\partial z} &= -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}\\ \frac{\partial i(z,t)}{\partial z} &= -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t}\end{aligned}}$$

Under steady-state conditions (suppressing the time factor)…

$$\frac{\partial v(z)}{\partial z} = -(R+j\omega L)\,i(z) \tag{1}$$

$$\frac{\partial i(z)}{\partial z} = -(G + j\omega C)\,v(z) \tag{2}$$

Rewritting equation (2),

$v(z) = -\dfrac{1}{(G + j\omega C)}\dfrac{\partial i(z)}{\partial z}$ and substituting this result back into equation (1):

$$\frac{\partial}{\partial z}\left(-\frac{1}{(G + j\omega C)}\right)\frac{\partial i(z)}{\partial z} = -(R+j\omega L)\,i(z)$$

$$\frac{\partial^2 i(z)}{\partial z^2} = (R+j\omega L)(G + j\omega C)\,i(z)$$

and similarly,

$$\frac{\partial^2 v(z)}{\partial z^2} = (R+j\omega L)(G + j\omega C)\,v(z)$$

Let

$$\boxed{\gamma = \alpha + j\beta \equiv \sqrt{(R + j\omega L)(G + j\omega C)}}$$

$\gamma \equiv$ complex propagation constant
$\alpha \equiv$ attenuation constant [Np/m]
$\beta \equiv$ phase constant [rad/m]

Using the propagation constant we uncover wave equations.

$$\boxed{\begin{aligned}\frac{\partial^2 i(z)}{\partial z^2} - \gamma^2 i(z) = 0 \\ \frac{\partial^2 v(z)}{\partial z^2} - \gamma^2 v(z) = 0\end{aligned}}$$

The solutions to the wave equations come in the form of traveling waves.

$$\boxed{v(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}} \tag{3}$$

$$\boxed{i(z) = I_o^+ e^{-\gamma z} + I_o^- e^{+\gamma z}} \tag{4}$$

$$e^{-\gamma z} \rightarrow +z \text{ directed wave} \qquad\qquad e^{+\gamma z} \rightarrow -z \text{ directed wave}$$

Substituting equation (3) into equation (1) …

$$-\gamma V_o^+ e^{-\gamma z} + \gamma V_o^- e^{+\gamma z} = -(R + j\omega L)\, i(z)$$

$$i(z) = \frac{\gamma}{(R + j\omega L)} \left(V_o^+ e^{-\gamma z} - V_o^- e^{+\gamma z} \right) \tag{5}$$

In comparing equations (4) and (5), we find that we can define a characteristic impedance.

$$Z_o = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-}$$

$$\boxed{Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}}} \tag{6}$$

Thus our traveling wave expressions are as follows.

$$\boxed{v(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}}$$

$$\boxed{i(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{+\gamma z}}$$

The propagation constant (γ) and characteristic impedance (Z_o) are two fundamental properties that define the behavior of transmission lines. Both depend on R, L, G and C of our lumped element model that in turn depend on the geometry of the transmission line and the relevant material properties.

Material Properties:

$$\varepsilon = \varepsilon_r \varepsilon_o = (1 + \chi_e)\varepsilon_o$$

χ_e electric susceptibility
materia property

$\chi_e = \varepsilon_r - 1$ (in Vacume $\chi_e = 0$)

$\vec{P} = \varepsilon_o \chi_e \vec{E}$ polarization of the material

$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = \varepsilon_o (1 + \chi_e)\vec{E} = \varepsilon_r \varepsilon_o \vec{E}$

$\vec{P}(t) = \varepsilon_o \int_{-\infty}^{t} \chi_e(t-t_o)\, \vec{E}(t_o)\, dt_o$ convolution.

$\mathcal{F}\left(\vec{P}(t)\right) = \varepsilon_o \chi_e(\omega) \vec{E}(\omega) = \vec{P}(\omega)$

$\chi_e(\omega)$ is frequency dependent

→ this results in dispersion.

$\Rightarrow$ {the material can't react instantaneously to $\vec{E}$}

Material Properties Relevant To Transmission Lines

Permittivity

When an applied electric field interacts with a material, the field tends to polarize (align) the atoms or molecules of the material. The 'permittivity' of a material defines the extent to which this polarization occurs. In general, a material's permittivity is a complex quantity, the imaginary portion (ε'') of which is a measure of a material's loss behavior. That is,

$$\varepsilon \equiv \text{ permittivity} = \varepsilon' - j\varepsilon''$$

In the field of microwaves, a dielectric's loss is more commonly given by its 'loss tangent' ($\tan\delta$) which includes both damping of the vibrating dipole moments and conductive loss.

$$\varepsilon = \varepsilon'\left(1\text{-}j\tan\delta\right)$$

The permittivity of free space (considered lossless) is:

$$\varepsilon_o \equiv \text{ permittivity of free space} = 8.854\times10^{-12}\text{ F/m .}$$

To simplify matters, when quoting the real part of the permittivity of a material (its 'dielectric constant'), we normalize the value to that of free space and thus quote the material's 'relative dielectric constant' (ε_r).

$$\varepsilon_r \equiv \text{ relative dielectric constant} = \frac{\varepsilon'}{\varepsilon_o}$$

Example values around 10 GHz

Material	ε_r	**tanδ**	**Frequency**
Teflon	2.08	0.0004	10 GHz
Al2O3	9.5-10	0.0003	10 GHz
Silicon	11.7-11.9	0.004	10 GHz
GaAs	13	0.006	10 GHz
Styrofoam	1.03	0.0001	3 GHz
Distilled water	76.7	0.157	3 GHz

Note: These values are frequency dependent; tanδ values depend on resisitivity. For example the tanδ value quoted for Si is based on high-resistivity material, not CMOS Si!

Permeability

In a similar fashion to the permittivity, we need a measure of the extent to which a material is influenced by a magnetic field. The 'permeability' of a material describes the ability of a magnetic field to align the 'magnetic dipoles' within the material. The permeability of free space is

$$\mu_o \equiv \text{ permeability of free space} = 4\pi\times10^{-7} \text{ H/m}.$$

While magnetic materials find application in microwave systems (phase shifters and circulators are example microwave components exploiting magnetic behavior), we will confine ourselves in EE 433 to non-magnetic materials and take the relative permeability (the analog to relative dielectric constant) to be unity. That is, $\mu_r = 1$.

Conductivity

While permittivity and permeability are used to describe materials that are poor electrical conductors (i.e. insulators), a material that readily conducts electrical current is characterized by its 'conductivity' (σ) which is often given in $1/(\Omega\text{-m})$. The conductivity values of several metals are given below.

Example Conductivity Values

Material	**Conductivity (S/m)**
Aluminum	$3.816\text{x}10^7$
Gold	$4.098\text{x}10^7$
Copper	$5.618\text{x}10^7$
Silver	$6.173\text{x}10^7$
Solder	$7.0\text{x}10^6$

Note: These are typical values for the given materials.

Resistivity

Resistivity is simply the inverse of the conductivity. The units for resistivity are thus resistance-length, most often quoted in (Ω-cm).

Skin Depth

High frequency EM fields do not penetrate conductors very far beneath the conductor surface. The 'skin depth' indicates the depth at which the amplitude of the field decays to 1/e (to ~ 37% of its original value). The skin depth is given by:

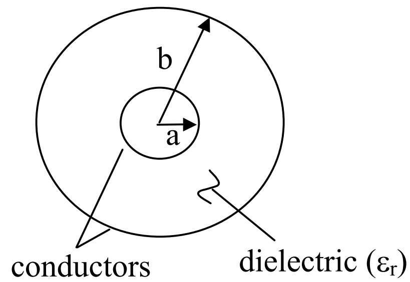
$$\delta_s \equiv \text{ skin depth} = \frac{1}{\alpha} = \sqrt{\frac{1}{\pi f \mu \sigma}} = \frac{1}{20\pi}\sqrt{\frac{1}{f\left[GHz\right]\sigma}}$$

For example, the skin depth of gold at 6 GHz is approximately 1 μm. As a rule of thumb, one strives to have a metal thickness in a planar circuit that is a few skin depths deep. Therefore, a gold thickness of a few microns at 6 GHz should be sufficient!

<end lecture 4>

Transmission Line Parameters for a Simple Coaxial Line

Consider the following cross section of a coaxial cable, recalling that the EM field is confined to the dielectric region.



The lumped element values for the distributed line parameters of a coax may be derived from electrostatic considerations (see standard EM texts) and are given as follows.

$$L = \frac{\mu'}{2\pi}\ln\frac{b}{a} \qquad C = \frac{2\pi\varepsilon'}{\ln\left(\frac{b}{a}\right)} \qquad R = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right) \qquad G = \frac{2\pi\omega\varepsilon''}{\ln\left(\frac{b}{a}\right)}$$

where

$$R_s \equiv \text{surface resistivity} = \frac{1}{\sigma\delta_s} \quad [\Omega]$$

A few things to note…

· Both geometry (a and b) and materials properties (μ, ε) play a role in defining the line parameters.
· L and C are lossless, whereas R and G introduce loss.

In general, we will deal with practical TLs, that is, TLs that exhibit low loss. If we assume that the metal used in the transmission line has infinite conductivity ($\sigma \to \infty$, Perfect Electrical Conductor -- PEC) and that the dielectric used is lossless ($\varepsilon'' = 0$), from equation 6 we find that the characteristic impedance of a coax is given by:

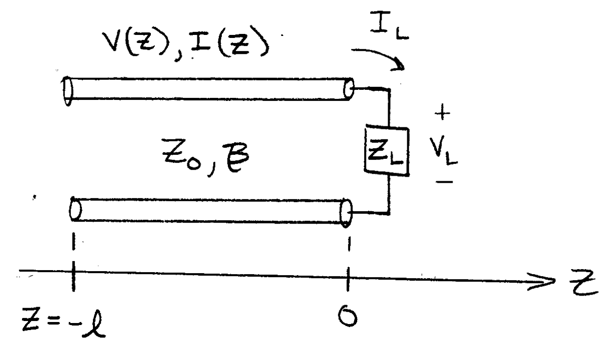
$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\ln\frac{b}{a}\sqrt{\frac{\mu'}{\varepsilon'}}\ .$$

Again in the lossless case, the propagation constant of a coax is purely imaginary, resulting in a phase constant solely dependent on material parameters as given by:

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\frac{\mu'}{2\pi}\ln\frac{b}{a}\frac{2\pi\varepsilon'}{\ln\frac{b}{a}}} = \omega\sqrt{\mu'\varepsilon'}$$

Terminated TL's and Reflected Waves

Assume a lossless TL, terminated with a load of impedance Z_L.



- Assume there exists a source at $z<0$ which creates a forward traveling wave: $V_0^+ e^{-j\beta z}$ & $I_0^+ e^{-j\beta z}$
- We know that $Z_0 = \dfrac{V_0^+}{I_0^+}$

total voltage & current

- At the load $(z=0)$, $\dfrac{V(z=0)}{I(z=0)} = Z_L$

$\Rightarrow$ There must be a reflected wave emerging at the load if $Z_L \neq Z_0$!

- If $Z_L \neq Z_0$,

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

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Thus, at the load,

$$Z_L = \frac{V(z=0)}{I(z=0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$$

Solving for V_0^- ⟶ $V_0^- = \dfrac{Z_L - Z_0}{Z_L + Z_0} V_0^+$

The voltage reflection coefficient at the load is defined as

$$\boxed{\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$ ← memorize

The total voltage and current may be written as

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L e^{j\beta z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma_L e^{j\beta z} \right]$$

Thus, the total voltage and current on the TL consis of a superposition of incident and reflected wave

↳ This situation creates standing waves!

Now let's consider the time-average power flow along the transmission line.

$$P_{AV} = \frac{1}{2} Re\left[V(z) I^*(z)\right]$$

$$= \frac{1}{2} Re\left[V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) \frac{V_0^{+*}}{Z_0} \left(e^{j\beta z} - \Gamma^* e^{-j\beta z}\right)\right]$$

$$= \frac{1}{2} \frac{|V_0^+|^2}{Z_0} Re\left\{1 - \Gamma^* e^{-j2\beta z} + \Gamma e^{j2\beta z} - |\Gamma|^2\right\}$$

$$= \frac{1}{2} \frac{|V_0^+|^2}{Z_0} Re\left\{1 - |\Gamma|^2 + 2j\, Im\left(\Gamma e^{j2\beta z}\right)\right\}$$

And finally,

$$P_{AV} = \frac{1}{2} \frac{|V_0^+|^2}{Z_0} \left(1 - |\Gamma|^2\right)$$

→ The average power flow is constant along the lossless line.

NOTE:

$$P_{AV} = \underbrace{\frac{1}{2} \frac{|V_0^+|^2}{Z_0}}_{\text{incident power}} - \underbrace{\frac{1}{2} \frac{|V_0^+|^2}{Z_0} |\Gamma|^2}_{\text{reflected power}}$$

From the previous expression we see,

if $\Gamma = 0 \rightarrow$ maximum power is delivered to Z_L

if $\Gamma = 1 \rightarrow$ zero average power is delivered to Z_L

* The "Return Loss" (RL) is defined as

$$\boxed{RL = -20 \log |\Gamma| \quad [dB]}$$

if $|\Gamma| = 0 \rightarrow RL = \infty$ dB

if $|\Gamma| = 1 \rightarrow RL = 0$ dB

When $\Gamma \neq 0$, a standing wave exists on the TL

Now
$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L e^{j\beta z} \right]$$
$$= V_0^+ \left[1 + \Gamma_L e^{j2\beta z} \right] e^{-j\beta z}$$

$$\Rightarrow |V(z)| = |V_0^+| \left| 1 + \Gamma_L e^{j2\beta z} \right|$$

From the previous figure $z = -\ell$

$$\Rightarrow |V(z=-\ell)| = |V_0^+| \left| 1 + \Gamma_L e^{-j2\beta \ell} \right|$$

<end lecture 5>

Terminated Transmission Lines

Constants, etc.

$f := 3 \cdot 10^9$ (3 GHz) $\quad c := 3 \cdot 10^8$ (m/s) $\quad j := \sqrt{-1}$

$$\lambda := \frac{c}{f} \qquad \beta := \frac{2 \cdot \pi \cdot f}{c}$$

Transmission Line Characteristic Impedance: $Z_o := 1$

Magnitude of the forward traveling wave: $V_{o\_plus} := 1$

As an example, let's take the load reflection coefficient to be: $\Gamma_L := 0.5 + j \cdot 0.3$

$$\theta := \arg(\Gamma_L)$$

Thus,

$$|\Gamma_L| = 0.583 \qquad \theta \cdot \frac{180}{\pi} = 30.964$$

In the general case, the magnitude of the voltage and current along a terminated transmission line may be expressed as follows:

$$V_{mag}(l) := |V_{o\_plus}| \cdot \left|1 + |\Gamma_L|\, e^{j \cdot (\theta - 2 \cdot \beta \cdot l)}\right|$$

$$I_{mag}(l) := \frac{|V_{o\_plus}|}{Z_o} \cdot \left|1 - |\Gamma_L| \cdot e^{j \cdot (\theta - 2 \cdot \beta \cdot l)}\right|$$

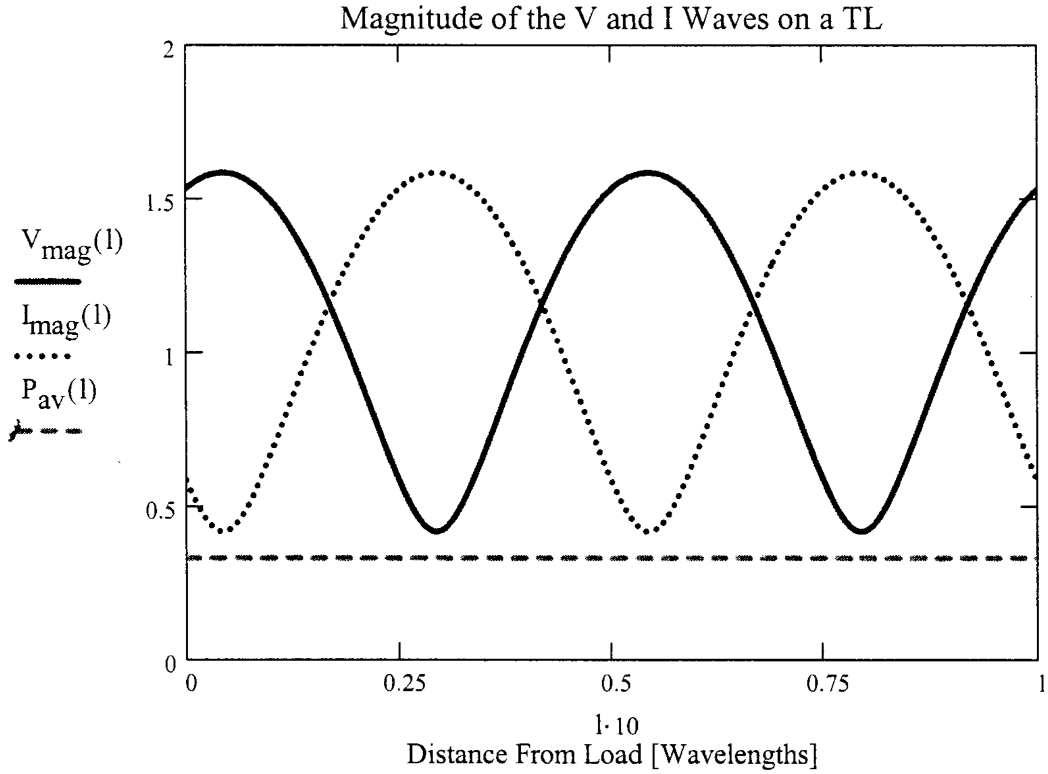
In the above expressions, "l" is the distance from the load to the point of interest on the TL.

$l := 0, 0.00001 .. 0.1$ [m]

NOTE: The TL is one wavelength long.

The average power:

$$P_{av}(l) := \frac{1}{2} \cdot \frac{(|V_{o\_plus}|)^2}{Z_o} - \frac{1}{2} \cdot \frac{(|V_{o\_plus}|)^2}{Z_o} \cdot (|\Gamma_L|)^2$$



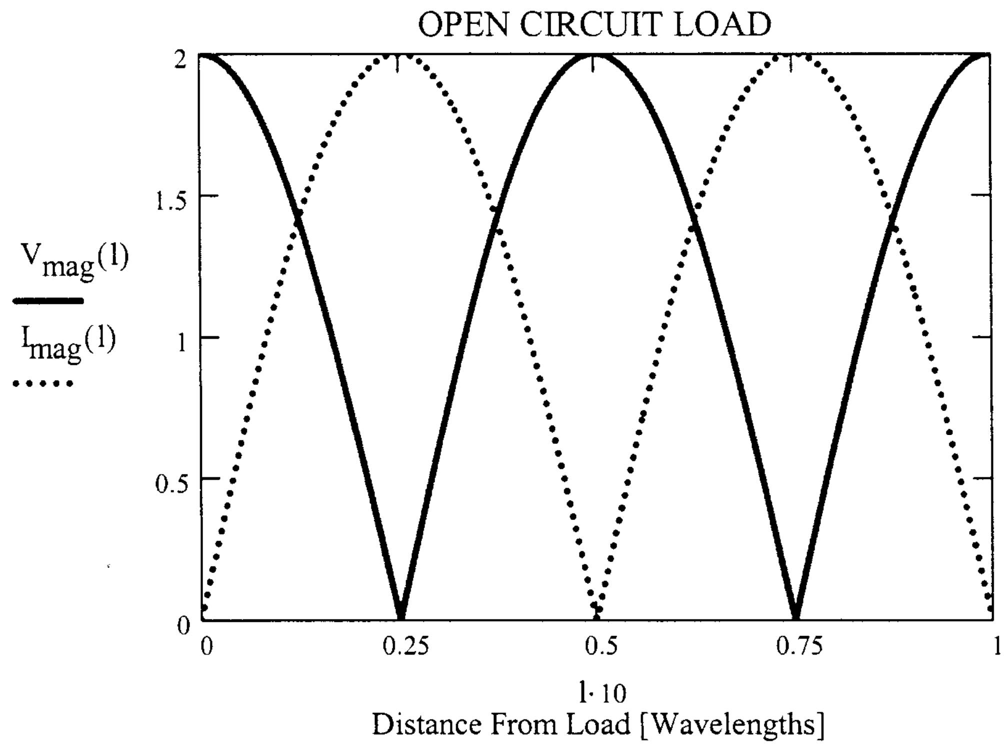
Voltage and Current form standing waves -- magnitude oscillates with position. The average power stays constant.

Special cases of the terminated transmission line...

I. OPEN CIRCUIT $\quad \Gamma_L := 1 \qquad \theta := \arg(\Gamma_L)$

$$V_{mag}(l) := |V_{o\_plus}| \cdot \left| 1 + |\Gamma_L| \cdot e^{j \cdot (\theta - 2 \cdot \beta \cdot l)} \right|$$

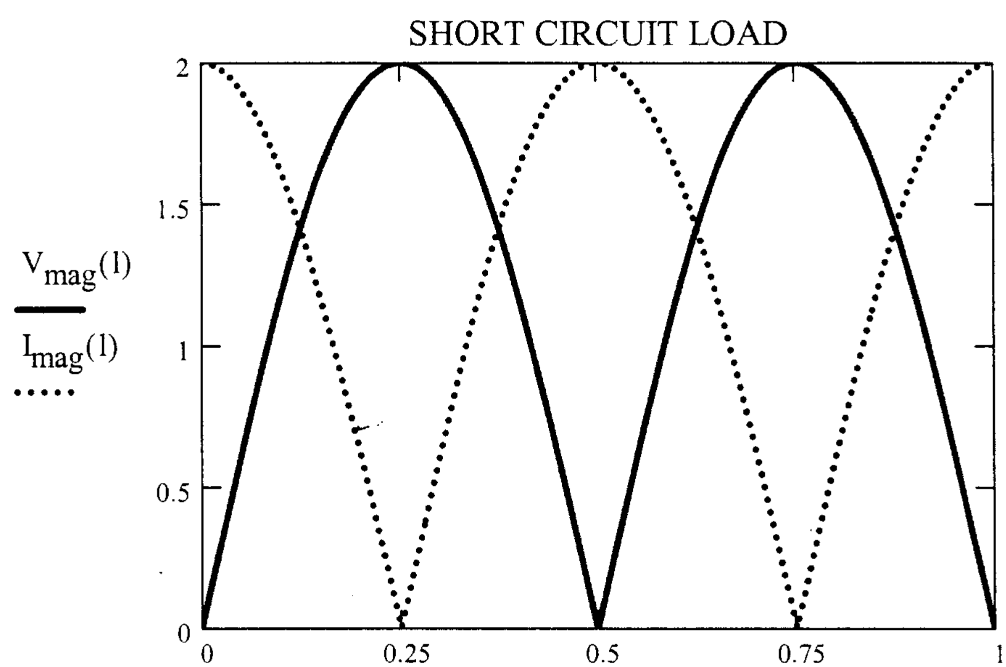
$$I_{mag}(l) := \frac{|V_{o\_plus}|}{Z_o} \cdot \left| 1 - |\Gamma_L| \cdot e^{j \cdot (\theta - 2 \cdot \beta \cdot l)} \right|$$



II. SHORT CIRCUIT $\quad \Gamma_L := -1 \quad \theta := \arg(\Gamma_L)$

$$V_{mag}(l) := \left|V_{o\_plus}\right| \cdot \left|1 + \left|\Gamma_L\right| \cdot e^{j \cdot (\theta - 2 \cdot \beta \cdot l)}\right|$$

$$I_{mag}(l) := \frac{\left|V_{o\_plus}\right|}{Z_o} \cdot \left|1 - \left|\Gamma_L\right| \cdot e^{j \cdot (\theta - 2 \cdot \beta \cdot l)}\right|$$



- In general, Γ_L is complex $\rightarrow$ $\Gamma_L = |\Gamma| e^{j\theta}$

$$\Rightarrow \boxed{|V(z=-\ell)| = |V_0^+| \left| 1 + |\Gamma_L| e^{j(\theta - 2\beta\ell)} \right|}$$

- Maximum occurs when $e^{j(\theta-2\beta\ell)} = 1$
  i.e. when $\theta - 2\beta\ell = 0, 2\pi, 4\pi, \ldots$ $\Big\}$ $V_{max} = |V_0^+|(1+|\Gamma_L|)$
- Minimum occurs when $e^{j(\theta-2\beta\ell)} = -1$
  i.e. when $\theta - 2\beta\ell = \pi, 3\pi, 5\pi, \ldots$ $\Big\}$ $V_{min} = |V_0^+|(1-|\Gamma_L|)$

A measure of the mismatch on a TL, the standing wave ratio (SWR), is given by

$$SWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

AKA VSWR (Voltage Standing wave ratio)

$$1 < SWR < \infty$$

matched load

load an open or a short (total reflection)

Distance between maxima: $\theta - 2\beta\ell = 2\pi$

$\theta \rightarrow$ constant $\quad \beta\ell = \pi \quad \beta = \frac{2\pi}{\lambda} \quad \frac{2\pi}{\lambda}\ell = \pi$

$$\Rightarrow \ell = \frac{\lambda}{2}$$

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Recall that the average power flow is constant at any point along a lossless TL. We have just discovered that the voltage oscillates on a mismatched line. Taken together, these two facts suggest that the impedance looking down the line will vary with position on a mismatched line!

At a distance $\ell = -z$ from the load, the input impedance seen looking toward the load is,

$$Z_{in} = \frac{V(-\ell)}{I(-\ell)} = \frac{V_0^+\left[e^{j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right]}{V_0^+\left[e^{j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right]} Z_0 = \frac{1+\Gamma_L e^{-j2\beta\ell}}{1-\Gamma_L e^{-j2\beta\ell}} Z_0$$

Also, $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$$\rightarrow \boxed{Z_{in}(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell}}$$

The TL Impedance Equation

<end lecture 6>