

Location of Cardinal Points from the ABCD Matrix for the General Optical System

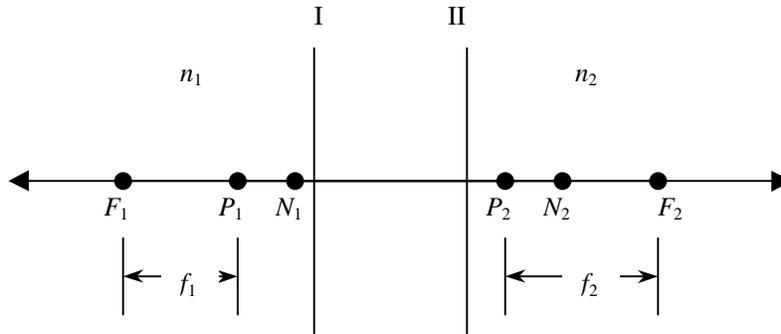


Figure 1 Cardinal points of a system which is characterized by an ABCD matrix between an input plane I and an output plane II.

| Cardinal Point | Measured | | Function | Special Case |
|------------------------|----------|-------|---------------------|--------------|
| | From | To | | |
| First Focal Point | P_1 | F_1 | $-n_1/(n_2C)$ | $-1/C$ |
| First Principle Point | I | P_1 | $(n_1-n_2D)/(n_2C)$ | $(1-D)/C$ |
| First Nodal Point | I | N_1 | $(1-D)/C$ | $(1-D)/C$ |
| Second Focal Point | P_2 | F_2 | $-1/C$ | $-1/C$ |
| Second Principle Point | II | P_2 | $(1-A)/C$ | $(1-A)/C$ |
| Second Nodal Point | II | N_2 | $(n_1-n_2A)/(n_2C)$ | $(1-A)/C$ |

Table 1

When analyzing an optical system, we often are first interested in basic properties of the system such as the focal length (or *optical power* of the system) and the location of the focal planes, etc. In the limit of paraxial optics, any optical system, from a simple lens to a combination of lenses, mirrors and ducts, can be described in terms of six special surfaces, called the cardinal surfaces of the system. In paraxial optics, these surfaces are planes, normal to the optical axis. The point where the plane intersects the optical axis is the cardinal point corresponding to that cardinal plane. The six special planes are:

- | | |
|-----------------------|------------------------|
| First Focal Plane | Second Focal Plane |
| First Principle Plane | Second Principle Plane |
| First Nodal Plane | Second Nodal Plane |

Once we know the location of these special planes, relative to the physical location of the lenses in the system, we can describe all of the paraxial imaging properties of the system.

First Focal Plane: The first focal plane is in the left hand space of the optical system, which by our convention corresponds to the object space of the imaging system. The intersection of the first focal plane with the optical axis is the first focal point F_1 . Rays originating from this point will leave the system parallel to the optical axis on the right side, focused to infinity in the image space. The distance between the first focal plane and the first principle plane, F_1P_1 , is the first focal length of the system, f_1 .

Second Focal Plane: The second focal plane is in the right hand, or image space of the optical system. The intersection of the second focal plane with the optical axis is the second focal point, F_2 . Rays coming from infinity on the left, and parallel to the optical axis, will come to a focus at F_2 . The distance from the second principle plane to the second focal plane, P_2F_2 , is the second focal length of the system, f_2 . The ratio of the first and second focal lengths of the system is given by $|f_1/f_2|=n_1/n_2$, where n_1 and n_2 are the indices of refraction in the object and image spaces, respectively.

First Principle Plane: The first principle plane is in the left hand, object space of the system. The point where the first principle plane intersects the optical axis is the first principle point, P_1 .

Second Principle Plane: The second principle plane is in the right hand, image space of the system. The point where the second principle plane intersects the optical axis is the second principle point, P_2 .

The first and second principle planes are conjugate planes (meaning that a point in the first principle plane is imaged by the system to a point in the second principle plane) with the special property of unity magnification. Any ray crossing the first principle plane at height y_1 will cross the second principle plane at height $y_2=y_1$.

Nodal Planes: The nodal points N_1 and N_2 are conjugate axial points with the special property that a ray through N_1 is conjugate to a parallel ray through N_2 . In other words, the nodal points are conjugate points in the object and image space with unity angular magnification. The planes normal to the axis that contain the nodal points are called the nodal planes. If $n_1=n_2$ then the nodal planes coincide with the principle planes.

The usefulness of the cardinal points is best illustrated with some specific examples. But first, consider any arbitrary optical system for which we are given the location of the principle planes and the focal points. How can we trace a paraxial ray through this system, and locate the image of a particular object point? Typically, for a thin lens, one traces rays using the following simple construction, illustrated in Figure 2. Consider an object space ray AB . We can construct a ray parallel to AB but going through the center of the lens where it does not deviate from its initial path. We extend this ray to the point where it intersects the second focal plane. Since parallel rays on the object side must meet in the second focal plane, we can now draw the image space ray that is conjugate to the object ray AB . This ray must pass through the same point in the second focal plane as the parallel ray.

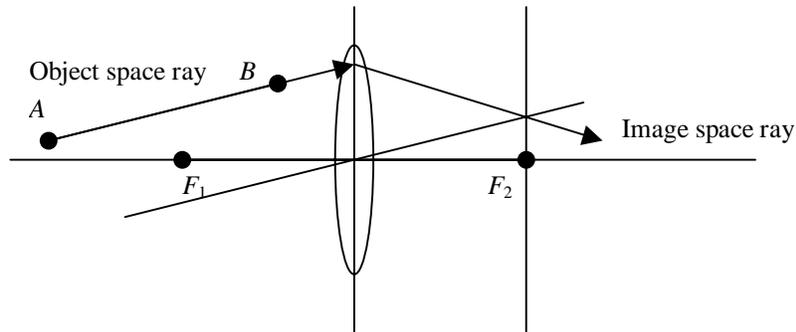


Figure 2

To find the image of a point, a similar construction is used. Consider the object point Q_1 in Figure 3. We can locate the image by constructing two special rays through the lens. First, we can extend a ray from Q_1 parallel to the optical axis. After passing through the lens this ray must go through the second focal point, F_2 . A second ray is drawn from Q_1 through the first focal point F_1 to the lens. This ray must exit the system parallel to the axis, since it goes through the focal point. The intersection of these two rays locates the image point Q_2 . A third ray may be drawn from Q_1 through the lens center. The undeviated ray must also pass through the image point Q_2 .

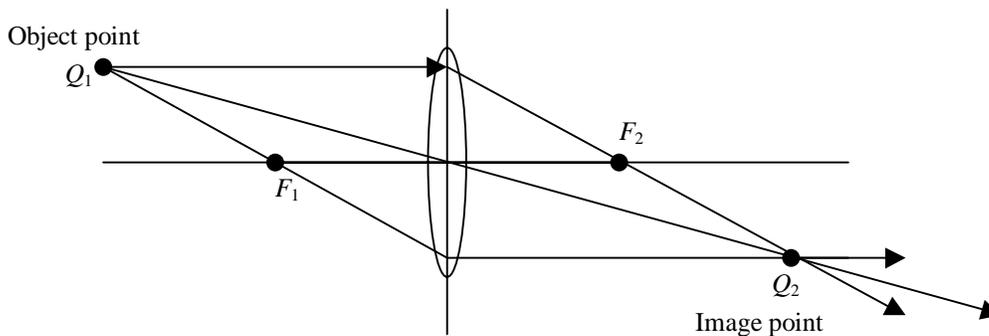


Figure 3

Now consider the more general system, specified only by the locations of the cardinal points. The same technique that is used for the thin lens is adopted, but now instead of drawing rays to the lens, we draw them to the principle planes. Beginning at point Q_1 , draw a ray parallel to the optical axis until it intersects the first principle plane. Because the magnification between principle planes is unity, this ray will emerge from the second principle plane at the same height. The two rays are connected with a dashed line to show that they are conjugate to one another. Because this object ray was parallel to the axis to begin with, the dashed line looks like an extension of the ray, so we often talk about drawing a ray parallel to the axis from Q_1 to the *second* principle plane. From the point where the ray emerges from the second principle plane, we draw the image space ray through the second focal point F_2 . We construct a second ray from Q_1 through the first focal point F_1 , extending the ray until it intersects the first principle plane. Again we transfer this ray through the system to the second principle plane preserving unity magnification between the two (shown again as a dashed line parallel to the axis). The

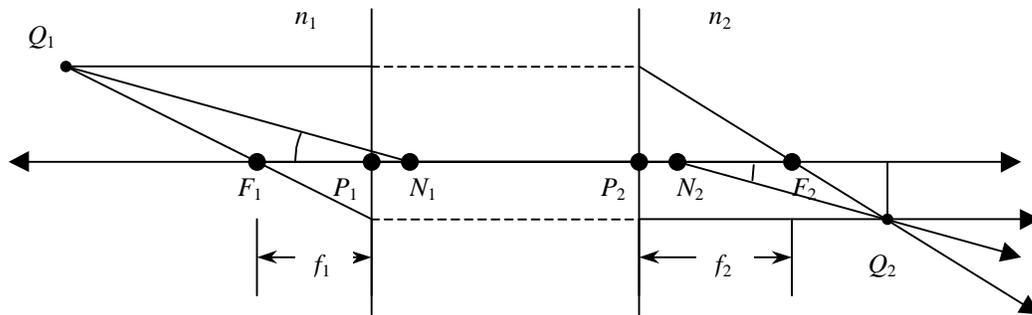
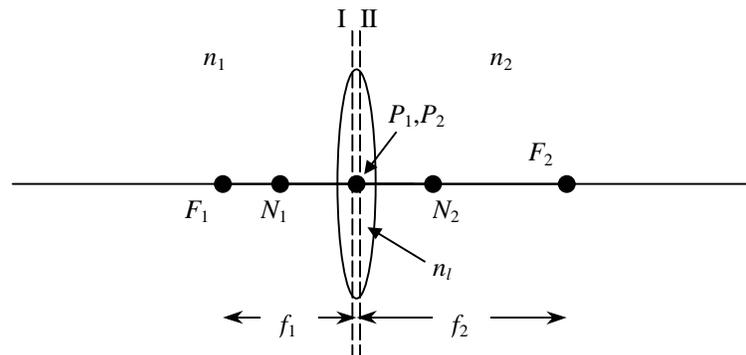


Figure 4

ray emerging must travel parallel to the axis on the image side since it passed through the first focal point. The intersection of these two rays must coincide with the image point Q_2 . Again, a third ray may be constructed by using the unity angle magnification property of the nodal points. A ray drawn from Q_1 to the first nodal point N_1 will emerge from the second nodal point N_2 along a parallel direction. This ray too must intersect the image point Q_2 .

Consider some specific examples.

Thin Lens



The ABCD matrix for the thin lens is $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & \frac{n_1}{n_2} \end{bmatrix}$, where

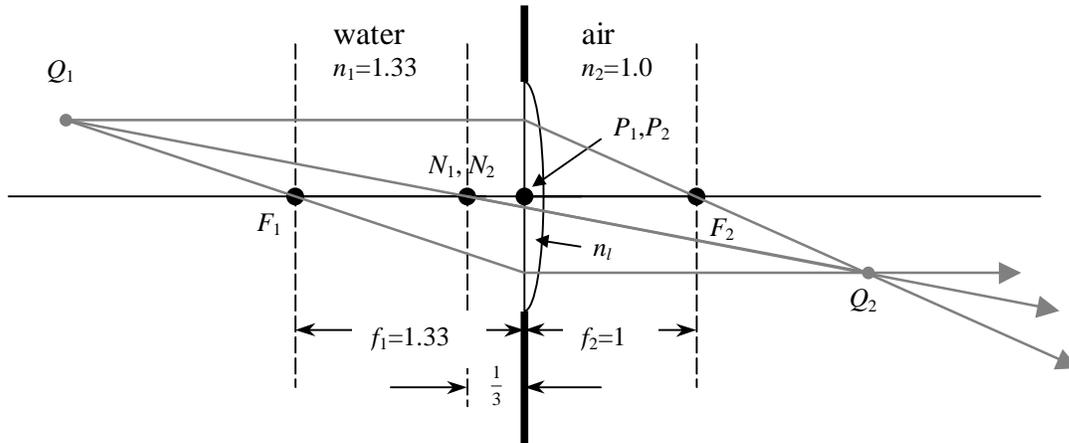
$-\frac{1}{f_2} = -\frac{n_2 - n_l}{n_2 R_2} - \frac{n_l - n_1}{n_2 R_1}$. The input and output planes for this system are both located

right at the lens. Referring to Table 1, we see that the first and second principle planes are also located at the lens. We have specified the ABCD matrix in terms of the second focal length f_2 , which is given above. From the table we see that the first focal length is given by $f_1 = n_1 f_2 / n_2$. We define the optical power P of the system as $P = n_1 / f_1 = n_2 / f_2$, so

the power of a thin lens is given by $P = \frac{n_2 - n_l}{R_2} + \frac{n_l - n_1}{R_1}$. From Table 1 we can also

find the location of the nodal points. We see that N_1 is located $(f_1 - f_2)$ in front of the lens, and N_2 is located $(f_2 - f_1)$ behind the lens.

As an example, consider a plano-convex thin lens with $f = 1$ in air. For the plano-convex lens, $R_1 = \infty$ so all of the power of the lens is due to the second interface. We plan to use this lens as a “water immersion” lens to image something in water and form an image in air. The situation is depicted in the following drawing.



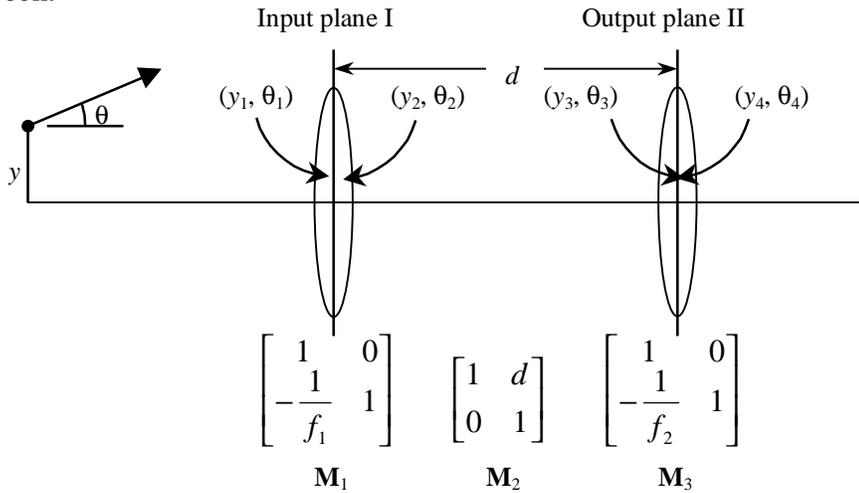
We can locate the principle planes and nodal points using the results just presented. We assume that $n_1 = 1.33$ for water, and $n_2 = 1.0$ for air. The focal length on the air side is the same as when the lens was in air (since $R_1 = \infty$), so $f_2 = 1$. The focal length on the water side is $f_1 = (n_1/n_2) f_2 = 1.33$. The first nodal point is located $f_1 - f_2 = 1/3$ in front of the lens. The second nodal point is located $f_2 - f_1 = -1/3$ behind the lens, or in other words coincident with the first nodal point. Typical rays for an object at Q_1 and image at Q_2 have been sketched on the diagram.

The power of the matrix method becomes evident for more complicated optical systems. Even a relatively “simple” optical system consisting of only a few lenses can become cumbersome when one must sequentially relay the image from one optical surface to the next. Furthermore, the overall performance of the system is often more apparent when the system is represented as its ABCD matrix. In the next example we will consider the properties of a two lens system by analyzing its ABCD matrix.

Two Thin Lenses

Consider now a system composed of two thin lenses in air, with focal lengths f_1 and f_2 respectively, and separated by a distance $d = f_1 + f_2$. (When considering multiple lenses or optical surfaces we will use the notation that a single subscript refers to the specific lens or optical surface. If the indices of refraction on either side of the surface or lens are different, then we will use a double subscript, with the first number referring to the number of the surface and the second subscript referring to either the object or image space, e.g. f_{12} refers to the second focal distance of lens “1”.) To write the matrix for the

combination of the two lenses we first write the matrices for each lens and the space in between.



A ray at any point in the system is represented by the height above the optical axis, y , and the angle the ray makes with the axis, θ . These two parameters are represented as a vector $\mathbf{r} = \begin{bmatrix} y \\ \Theta \end{bmatrix} = (y, \Theta)$. If $\mathbf{r}_1 = \begin{bmatrix} y_1 \\ \Theta_1 \end{bmatrix}$ is the ray vector just to the left of

the lens, then $\mathbf{r}_2 = \begin{bmatrix} y_2 \\ \Theta_2 \end{bmatrix}$ is the ray vector just to the right of the lens, where

$$\mathbf{r}_2 = \mathbf{M}_1 \mathbf{r}_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \Theta_1 \end{bmatrix}. \text{ Similarly, } \mathbf{r}_3 = \mathbf{M}_2 \mathbf{r}_2 = \mathbf{M}_2 \mathbf{M}_1 \mathbf{r}_1, \text{ and}$$

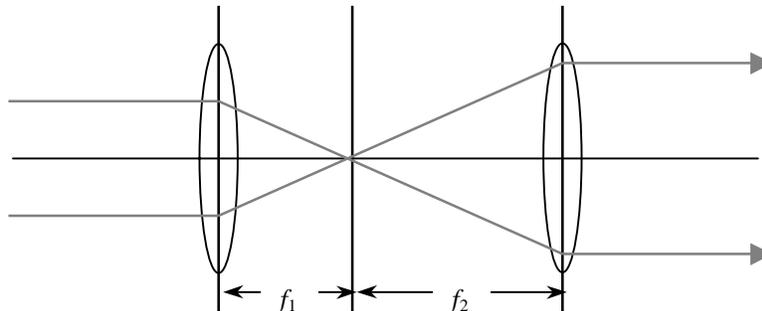
$\mathbf{r}_4 = \mathbf{M}_3 \mathbf{r}_3 = \mathbf{M}_3 \mathbf{M}_2 \mathbf{r}_2 = \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1 \mathbf{r}_1 = \mathbf{M}_{sys} \mathbf{r}_1$, where $\mathbf{M}_{sys} = \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1$ is the system matrix. For the two lens system above we can multiply out the matrices to obtain

$$\mathbf{M}_{sys} = \mathbf{M}_3 \mathbf{M}_2 \mathbf{M}_1 = \begin{bmatrix} 1 - \frac{d}{f_1} & d \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{d}{f_1 f_2} & 1 - \frac{d}{f_2} \end{bmatrix}.$$

Now we are ready to consider the problem of two lenses separated by $d=f_1+f_2$.

Substituting into \mathbf{M}_{sys} we obtain $\mathbf{M}_{sys} = \begin{bmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{bmatrix}$. Because $C=0$, the cardinal

points are all located at infinity. Furthermore, the focal length of the system is infinite, so that we call this system afocal, or telescopic. That means that rays parallel to the axis on



the object side are also parallel to the axis on the image side. With $C=0$, the output angle is given by $\theta = D\theta_1$, where $D = -f_1/f_2$ is the angular magnification. This arrangement can be used as a beam expander, with the lateral magnification given by $A = -f_2/f_1$.

The telescopic system can be used to form an image, even though the focal length of the system is infinite. Consider the above system, but now we consider the input plane to be a finite distance z_1 in front of the first lens, and the output plane to be a distance z_2 behind the last lens. Then the system matrix becomes

$$\mathbf{M}_{\text{sys}} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{f_2}{f_1} & f_1 + f_2 - \frac{1}{f_1 f_2} (z_1 f_2^2 + z_2 f_1^2) \\ 0 & -\frac{f_1}{f_2} \end{bmatrix}. \text{ In terms of}$$

the matrix elements, $y_{\text{out}} = Ay_{\text{in}} + B\theta_{\text{in}}$, so that if $B = 0$, the output position is independent of the input ray angle, which is the condition for imaging. In this case, an image is formed right at the output plane of the system. The magnification is given by the matrix element $A = -f_2/f_1$.

