Diagnose and Correct Mathematics Error Patterns

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Why learn to track and diagnose errors?

• Greatly strengthens feedback to student
• Individualizes the learning process
• Often can be used to identify student strengths on which to build
• By studying error patterns we can also research and apply best practices for eliminating the error
When looking at/for errors

• Focus on what a student does know.
• Try to plan instruction that would build on strengths.
• Consider if previous instruction could be source of error:
  • Do we as instructors overgeneralize or use weak vocab
Division Examples:

\[
\frac{-0.01035}{-0.05} = \frac{0.27}{5} \quad \frac{1.035}{0.35}
\]
Possible Solutions

Use lined paper turned 90 degrees.

Use algorithm that emphasizes place value

\[
\begin{align*}
5 \) 1.035 \\
1.0 \\
\underline{0.207} \\
0.35 \\
35 \\
\hline
0
\end{align*}
\]

\[
\begin{align*}
5 \) 1035 \\
200 \\
\underline{207} \\
7 \\
\underline{20} \\
0 \\
0 \underline{35} \\
35 \\
\underline{35} \\
0
\end{align*}
\]
Error diagnosis is not accidental.

• Some assessments must be made with error detection in mind.
• Previous example had a zero in the quotient.
  • Have taught out of textbooks with *none* of these anywhere in text.
  • Evaluated assessment devices with *none* of these.
  • Evaluated instructors who were confident students understood division, yet did not assess if students could deal with zero in the quotient.
• When detected, this error can be used to generate discussions that can grow knowledge outside of context of division—specifically place value.
Is this the same or different?

• Adding decimal may change why student made error
• Student may have thought decimal goes at the “end” in this problem which is a different misunderstanding. Would instruction be same?
Is this the same or different?

• Adding decimal may change why student made error
• Student knew to add a zero to continue the division.
• Would instruction be same?
Overgeneralization errors.

26) \((31.8)(0.05) = 1.59\)

\[
\begin{array}{c}
\text{318} \\
\text{x 5} \\
\hline
\text{1590}
\end{array}
\]

28) \((-0.01035) + (-0.05) = 0.0006207\)

\[
\begin{array}{c}
\text{1635} \\
\text{10} \\
\hline
\text{35}
\end{array}
\]
Possible solutions

• Avoid overgeneralization by highlighting both similarities and differences between multiplication and division.
  • Signs are dealt with similarly.
  • Decimal place value is not treated similarly.

• Write all divisions first in fraction notation so that factors of one can be both added (multiply to remove decimals) and removed (cancel/reduce).

\[
(\ -0.01035 \ ) \div (\ -0.05 \ ) = \frac{0.01035}{0.05} \cdot \frac{100}{100} = \frac{1.035}{5}
\]
How would you describe this error?

• Do you think it is common?

• How would you plan instruction?

• How does it appear in other topics/areas of algebra?
• Fairly common.
• Emphasize writing multiplication for exponents.
  • $2^3 = 2 \cdot 2 \cdot 2$
  • $2x^3 = 2 \cdot x \cdot x \cdot x$
  • $(2x)^3 = (2x)(2x)(2x)$
• And emphasize writing exponents for multiplication
  • $a \cdot c \cdot c = ac^2$
  • $a \cdot c \cdot a \cdot c = a^2c^2$
Does this student have the same error? Would you plan instruction the same?

Student from previous slide

Different student

\[ \frac{ac^2}{b} \cdot \frac{b}{ac} = \frac{ac}{b} \cdot \frac{ac}{b} \]
Subtle differences may lead to discussion or new question development

• When we observe:

\[
\frac{ac^2}{b} \cdot \frac{b}{ac}
\]

• We should create a new question.

\[
\frac{ac^2}{b} \cdot \frac{b}{a}
\]

(to see if student will cancel \(a\))
This error is more common in arithmetic and can hinder development in algebra.

The following is extremely common in developmental math:

- Same basic principle algebraically.
  - $ac^2 = a \cdot c \cdot c$
  - $2x^3 = 2 \cdot x \cdot x \cdot x$

- Students misunderstand effect of exponent.

- How can this be addressed?
Again, I would suggest factors.

• Have student write out all factors (especially factors of -1) for the following:

\[ -3^2 = -1 \cdot 3^2 = -1 \cdot 3 \cdot 3 \]

\[ (-3)^2 = (-1 \cdot 3)^2 = (-1 \cdot 3)(-1 \cdot 3) \]

Or consider the statement “the exponent only affects what it touches”.

\[ (-3)^2 = (\ ) (\ ) = (-3)(-3) \]
Of course some of our “help” leads to other student errors:

• Overheard an instructor say:

" $-3^2$ has one negative because the exponent does not include the sign so the answer is negative, but $(-3)^2$ has two negatives because of the exponent so it’s answer is positive”

• Sounds right don’t it....
Until the student did the next problem

\[ 2 - (-3)^2 = 2 + 3^2 \]

“but you just said two negatives is a positive in exponents”

At which point the conversation turned to order of operations.
What is the error here?
How to you plan instruction?

\[-3 + 2 [4 - (-6)]
\[-9 + 2 [10]
\[-7 [10]\]
Did you say the error is PEMDAS?

• Because I think that here PEMDAS is the error:

\[
12) \frac{2}{5} + \frac{1}{2} \cdot \frac{1}{7} = \frac{1 \cdot 1}{2 \cdot 7}
\]

\[
\frac{2}{3} \cdot \frac{14}{1} = \sqrt{\frac{28}{6}}
\]
Some data from January 2015

- Sample size of 60 students.
- Enrolled in Prealgebra at Montana State University
- Question posed on the fourth day of instruction to three different classes of 20.

\[12 \div 6 \cdot 2\]
Results tallied with no discussion/instruction allowed.

1\(\frac{2}{3}\)
“Discuss your results then vote on final answer.”
(still no instruction—just student interaction)
Note the description of the orange column.
By the first exam roughly 5 instructional days later:

\[ 15 \div 5 \cdot 3 \]

Was only missed by 2 of the same group of 60 students....

But struggles remained (45% incorrect) with the problem:

\[ 20 - 4(2 - 8) \]

partially because some still “did the parenthesis” and got:

\[ 20 - 4 - 6 \]

while most distributed a four to get:

\[ 20 - 8 - 32 \]
Both errors could have been prevented by a deeper understanding of order

- Parenthesis are not “done”
  - Except in the context of multiplication—which is third out of four in the sense of order—not first

- The operation inside of the grouping symbol is done, with the grouping symbol remaining in the work.

\[
20 - 4(2 - 8) \neq 20 - 4 - 6
\]

\[
20 - 4(2 - 8)
\]

Inside subtraction is first

\[
20 - 4(-6)
\]

Now the parenthesis do not denote order by rather multiplication...

\[
20 + 24
\]
Two suggestions:

• Teach the “rest” of the mnemonic device: PEMDASSDNKM
  Please
  Excuse
  My
  Dear
  Aunt
  Sally
  She
  Does
  Not
  Know
  Math

• Focus on the idea that there are only 4 steps to the order of operations and build understanding at the same time.
Prealgebra

1. Operations inside grouping symbols:
   • Radicals \(\sqrt{\text{______}}\)
   • Absolute Values | | 
   • Division, ______
   • { } or [ ] or ( )

2. Exponentials, roots, absolute values

3. Multiplication and division as they appear (left to right)

4. Addition and subtraction as they appear (left to right)
Fractions/signed numbers

1. Operations inside grouping symbols
2. Exponentials, roots, absolute values
3. Rewrite all divisions as multiplication by reciprocal, factor, cancel.
4. Rewrite all subtraction as addition of the opposite, collect like terms.

Algebra

1. Operations inside grouping symbols
2. Exponentials, roots, absolute values
3. Distribute.
4. Collect like terms.
Add to my dismay, old habits die hard:

• Despite 97% success on the exam a fellow instructor suggested the following problem (given within days of demonstrated success).

• A very significant number of students answered 1:

\[ 15 \div 5(3) \]

“because parentheses come first”
What about correct answers?

Simplify.
1) \( \frac{-14}{42} \) \( \frac{-4 \cdot 2}{7 \cdot 3} = \frac{-1}{3} \)

Simplify.
1) \( \frac{-14}{42} \div 7 = -\frac{2}{6} + \frac{2}{6} = \frac{1}{3} \)
First the simplify by factor problem.

• Should have separated sign from number factor: \(\frac{-1 \cdot 7 \cdot 2}{7 \cdot 2 \cdot 3}\)

• Student has had problems with examples like \(-3^2\) where separating sign from number factor would provide benefit: \(-3^2 = -1 \cdot 3^2\)

• Ignoring the error because of the correct answer is a disservice.
Simplify by division of top and bottom

Facts about this student

• Computationally strong but cannot do multistep problems.
• Has not progressed in algebra.
• Maintains current knowledge, but unable to make any gains.
• Attempts to solve all story problems with different combinations of multiplication, division, addition, and subtraction. Effectively demonstrating trial and error as only technique.
• When initially approached about the error was defensive and argumentative.
   After all she did get it correct.
Create a discussion with problems like:

- \(\frac{-14}{42} \div 7\) and write ones.

- Ask multiplication and use Addition:

Using work like:

\[
\frac{-14}{42} \div 7 = \frac{-14}{42} \div \frac{7}{1} = \frac{-14 \cdot 1}{42 \cdot 7}
\]

\[
= \frac{-1 \cdot 2 \cdot 7 \cdot 1}{2 \cdot 21 \cdot 7}
\]

\[
= \frac{-1 \cdot 2 \cdot 7 \cdot 1}{2 \cdot 21 \cdot 7}
\]

\[
= \frac{-14 \cdot 1}{42 \cdot 7}
\]

\[
\frac{3}{8} \cdot 4 = \frac{3}{8} \cdot \frac{3}{8} = \frac{12}{8}
\]
A different student had consistently failed and I had little, to no, success instructing.

Can you identify her errors? (there are two)

A. $12x - 5 + 2x = 10x - 5$
B. $-6 + 6x - 2 = -4 + 6x = -2x$
C. $8x + 2 - 6x = 2x + 2$
D. $10 + 4x - 6 = 4 + 4x = 8x$
Basic algebraic errors
(remember--all from same student)

I initially was distracted by the more common blue error I did not initially address the pattern of the less common red error.

A. $12x - 5 + 2x = 10x - 5$

B. $-6 + 6x - 2 = -4 + 6x = -2x$

C. $8x + 2 - 6x = 2x + 2$

D. $10 + 4x - 6 = 4 + 4x = 8x$
Plan for instruction based on error detection.

• Notice the high level of correct computation. Error was in reading the notation, but error was consistent in application.

• Began with color tiles to show like vs. unlike terms (addressed blue error before red)

• Had some difficulty doing signs and like terms at the same time, but once connection was made improvement was observed.

• Later discovered that this student was not a typical developmental student, but rather, much more accurately described as a first time learner.
Credit due
(especially for the diagnosis of the preceding example):