Intermediate Microeconomics
ECNS 301
Fall 2015

Homework #: 5

Due by the beginning of class on: Thursday October 22, 2015

Name: __________________________________________

Instructions:
There are 5 questions worth a total of 100 points. Answer each question clearly and concisely. You must show your work to receive credit. You are allowed to work with others, but all work must be your own.

Clearly print your name above and in the space provided on the next page. You must turn in both sides of this cover sheet along with your responses. You do not need to turn in the questions, only your responses with the cover sheet. All pages must be stapled to be graded.
Consumer Demand

1. Since 1900, real income has increased tremendously, yet the average number of children has fallen. Illustrate the following possible explanations in separate diagrams with budget sets and indifference curves between number of children and “all other goods”.

(a) Children are an inferior good; because we’re richer now we want fewer of them.

Solution: The increase in income shifts the budget constraint out from $B_1$ to $B_2$. The average number of children decreases from $c_1$ to $c_2$. 

![Diagram showing budget sets and indifference curves between number of children and "all other goods". The budget constraint shifts from $B_1$ to $B_2$, and the indifference curve shifts from $IC_1$ to $IC_2$. The number of children decreases from $c_1$ to $c_2$.](attachment:diagram.png)
(b) Children are a normal good, but it has become more expensive to bear and raise them.

**Solution:** There is an increase in income and an increase in the cost of raising children. The budget constraint changes from $B_1$ to $B_2$. The average number of children decreases from $c_1$ to $c_2$. 

![Graph showing the budget constraint and the change in children and other goods consumption.]
(c) Children are a normal good and they do not cost any more to raise than they did. However, preferences have changed, and couples today want smaller families than couples did in 1900.

**Solution:** The increase in income shifts the budget constraint out from $B_1$ to $B_2$. Since preferences have changed, the indifference curves can cross. The average number of children decreases from $c_1$ to $c_2$. 

![Graph showing budget constraints and indifference curves](attachment:graph.png)
2. A consumer's utility function is given by

\[ U(x, y) = 4x^{\frac{1}{2}} + y^{\frac{1}{2}} \]  

The consumer's income is \( I \), the price of good \( y \) is \( P_y \) and the price of good \( x \) is \( P_x \). (Warning: the algebra in this problem is messy but it is good practice.)

(a) What is the marginal rate of substitution?

**Solution:** The marginal rate of substitution is the ratio of the marginal utilities. The marginal utilities are

\[ \frac{\partial U}{\partial x} = 2x^{-\frac{1}{2}} \]
\[ \frac{\partial U}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}} \]

so the marginal rate of substitution is

\[ MRS = -\frac{\partial U}{\partial x} \frac{\partial U}{\partial y} = -2x^{-\frac{1}{2}} \cdot \frac{1}{2}y^{-\frac{1}{2}} = -4 \left( \frac{y}{x} \right)^{\frac{1}{2}}. \]

(b) What is the equation for the budget constraint?

**Solution:**

\[ P_x x + P_y y = I \]

(c) What is the demand function for \( x \) (as a function of prices and income)?

**Solution:** Setup the Lagrangian.

\[ \max_{x, y, \lambda} \mathcal{L}(x, y, \lambda) = 4x^{\frac{1}{2}} + y^{\frac{1}{2}} + \lambda(I - P_x x - P_y y) \]

Now take the first order conditions.

\[ \frac{\partial \mathcal{L}}{\partial x} = 2x^{-\frac{1}{2}} - \lambda P_x = 0 \]
\[ \frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}y^{-\frac{1}{2}} - \lambda P_y = 0 \]
\[ \frac{\partial \mathcal{L}}{\partial \lambda} = I - P_x x - P_y y = 0 \]
Solve for lambda in the first two equations, set them equal to one another, and then substitute this result back into the budget constraint.

\[
\lambda = \frac{2x^{-\frac{1}{2}}}{P_x}
\]

\[
\lambda = \frac{y^{-\frac{1}{2}}}{2P_y}
\]

\[
\frac{2x^{-\frac{1}{2}}}{P_x} = \frac{y^{-\frac{1}{2}}}{2P_y}
\]

\[
4P_y y^{\frac{1}{2}} = P_x x^{\frac{1}{2}}
\]

\[
y^{\frac{1}{2}} = \frac{P_x}{4P_y} x^{\frac{1}{2}}
\]

\[
y = \frac{P_x^2}{4P_y} x
\]

\[
I = P_x x + P_y y
\]

\[
I = P_x x + P_y \left( \frac{P_x^2}{4P_y} x \right)
\]

\[
I = x \left( P_x + \frac{P_x^2}{16P_y} \right)
\]

\[
x = \frac{I}{\left( P_x + \frac{P_x^2}{16P_y} \right)}
\]

This is the demand function for \(x\).
Applications of Consumer Demand

3. Consider Joe Blow’s utility function: \( U = xy \), where \( x \) is packs of cigarettes and \( y \) is dollars spent on all other goods. Joe has income \( m = 32 \); the price of cigarettes (\( x \)) is \( P_x = 1 \) and the price of all other goods (\( y \)) is \( P_y = 2 \).

(a) What is the optimal consumption bundle and what level of utility is achieved?

Solution: Setup the Lagrangian.
\[
\max_{x,y,\lambda} L(x, y, \lambda) = xy + \lambda(m - P_x x - P_y y)
\]

Now find the first order conditions and set them equal to zero.
\[
\begin{align*}
\frac{\partial L}{\partial x} &= y - \lambda P_x = 0 \\
\frac{\partial L}{\partial y} &= x - \lambda P_y = 0 \\
\frac{\partial L}{\partial \lambda} &= m - P_x x - P_y y = 0
\end{align*}
\]

Combine the first two first order conditions to get \( P_x x = P_y y \). Substitute into the last first order condition/budget constraint get get demands.
\[
\begin{align*}
m &= P_x x + P_y y \\
m &= P_x x + P_x x \\
x &= \frac{m}{2P_x} \\
y &= \frac{m}{2P_y}
\end{align*}
\]

The utility function is \( U = xy \). Substituting the demand functions into the utility function, we get
\[
U = \frac{m}{2P_x} \frac{m}{2P_y} = \frac{m^2}{4P_x P_y}
\]

Now let’s put the prices and income into the demand and utility functions.
\[
\begin{align*}
x &= \frac{m}{2P_x} = \frac{32}{2(1)} = 16 \\
y &= \frac{m}{2P_y} = \frac{32}{2(2)} = 8 \\
U &= \frac{m^2}{4P_x P_y} = \frac{32^2}{4(1)(2)} = 128
\end{align*}
\]
(b) Assume that the supply of cigarettes is perfectly elastic. The government imposes a tax of $1 per pack on cigarettes. What will be the after-tax price paid by Joe to consume a pack of cigarettes?

**Solution:** The after-tax price will be $2.

(c) What will the new optimal consumption bundle be with the tax and what is Joe's level of utility?

**Solution:** With $P_x = 2$, the quantity demanded of $y$ is still 8, but the quantity demanded of $x$ is now

$$x = \frac{m}{2P_x} = \frac{32}{2(2)} = 8.$$

Joe's level of utility is $U = (8)(8) = 64$.

(d) How much revenue will the government raise with the tax?

**Solution:** The government gets $1 per pack of cigarettes and 8 packs are bought. The tax revenue is $8.

(e) Consider two proposed tax reforms:

Reform 1: Eliminate the tax on cigarettes and impose a lump sum tax (effectively an income tax) that raises the amount of revenue equal to that of the $1 per unit tax.

Reform 2: Eliminate the tax on cigarettes and impose a lump sum tax that leaves the consumer at the level of utility reached under the $1 per unit tax.

i. By how much utility would Reform 1 make Joe better or worse off? Illustrate. (It will be helpful to draw your indifference curves and budget constraints to scale and make sure they pass through the correct points.)

**Solution:** The lump sum tax of $8 leaves the consumer with $m = 24$ in income. Now find demands and utility with $P_x = 1$ and $P_y = 2$.

$$x = \frac{m}{2P_x} = \frac{24}{2(1)} = 12$$

$$y = \frac{m}{2P_y} = \frac{24}{2(2)} = 6$$

$$U = \frac{m^2}{4P_xP_y} = \frac{24^2}{4(1)(2)} = 72$$

Reform 1 makes Joe better off since utility increases from 64 to 72.
ii. By how much would Reform 2 change the amount of revenue for the government? Illustrate. (Hints: To solve for the amount of the lump sum tax imposed in Reform 2, remember that the utility function given results in an even division of income between spending on $x$ and spending on $y$. Use a calculator for the messy calculations this question.)

**Solution:** Above we found utility to be

$$U = \frac{m^2}{4P_xP_y}.$$  

For Reform 2, we want to find the level of income such that $U = 64$, $P_x = 1$, and $P_y = 2$.

$$U = \frac{m^2}{4P_xP_y}$$

$$64 = \frac{m^2}{4(1)(2)}$$

$$m^2 = 512$$

$$m = 16\sqrt{2}$$
Instead of raising $8 in tax revenue, $16\sqrt{2} = 22.63$ of tax revenue is generated which is an increase of 14.63.

Note that the $y$ intercept changes from 16 to $8\sqrt{2}$. For both budget constraints, the $y$ intercept is equal to $m/p_y$. Since $p_y = 2$, multiply the $y$ intercept to get the income associated with each budget constraint. $B_1$ has an associated income level of 32 and $B_2$ has an associated income level of $16\sqrt{2}$. The difference in income is due to the lump sum tax.
4. The government subsidizes daycare for low income families (especially important now that welfare ends after 2 years). For simplicity, consider a single mother who wants to enroll in school. If possible, she would enroll in school full time and buy 40 hours of daycare. Assume all commodities are normal, indifference curves are smooth and convex, ignore any effects on the price of daycare, and ignore changes in her future income, etc. It is your job to evaluate the following three proposed government programs.

Plan A: Day care subsidies where the subsidized price is set so that exactly 40 hours of daycare are purchased.

Plan B: Direct cash payments which can be used for any purchase (welfare payments are continued while the family is in school). The family receives an exact quantity of additional income which leads to 40 hours of daycare purchased.

Plan C: The government provides 40 hours of day care free of charge to low income families. (Cost to government is current market price.)

(a) Show the income and substitution effects of Plan A graphically.
(b) Show the income effect of Plan B graphically. Is there a substitution effect?

**Solution:** There is no substitution effect associated with Plan B.
(c) In your graphs above, label the cost to the government of each plan. Which costs the government more: Plan A or Plan B?

**Solution:**

<table>
<thead>
<tr>
<th>Hours of Daycare</th>
<th>Other Goods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plan A</td>
<td>IC&lt;sub&gt;Plan A&lt;/sub&gt;</td>
</tr>
<tr>
<td>Plan B</td>
<td>IC&lt;sub&gt;Plan B&lt;/sub&gt;</td>
</tr>
<tr>
<td>IC&lt;sub&gt;1&lt;/sub&gt;</td>
<td></td>
</tr>
</tbody>
</table>

In the figure above, note points B, A and R on the vertical axis. The cost to the government for Plan A is $p_y(A - R)$ and the cost to the government for Plan B is $p_y(B - R)$.

Here’s why. Let $y$ be other goods and $x$ be hours of daycare. For Plan A, the price of $x$ decreases from $p_x$ to $p'_x$ due to the subsidy. Since the government lowered the price, they have to pay for the price difference for the 40 hours of daycare purchased and the cost to the government for Plan A is $(p_x - p'_x)40$.

For Plan B, the income increases from $m$ to $m'$ and the cost to the government is $m' - m$.

Points B, A and R are all on budget constraints, and each point has $x = 40$. We know that a budget constraint has the form $p_x x + p_y y = m$ and setting $x = 40$ and solving for $y$ we get

$$ y = \frac{m - p_x 40}{p_y}. $$

Point R is on the original budget constraint so

$$ D = \frac{m - p_x 40}{p_y}. $$
Point $A$ corresponds to Plan A where the price of $x$ is $p'_x$ and thus

$$A = \frac{m}{p_y} - \frac{p'_x}{p_y} 40.$$  

Point $B$ corresponds to Plan B where the income has increased from $m$ to $m'$ and thus

$$B = \frac{m'}{p_y} - \frac{p_x}{p_y} 40.$$  

Now the cost to the government for Plan A is $(p_x - p'_x)40$ and now we show that $(p_x - p'_x)40 = p_y(A - R)$. 

$$p_y(A - R) = p_y \left( \frac{m}{p_y} - \frac{p'_x}{p_y} 40 - \frac{m}{p_y} + \frac{p_x}{p_y} 40 \right)$$  

$$p_y(A - R) = p_y \left( \frac{p_x}{p_y} 40 - \frac{p'_x}{p_y} 40 \right)$$  

$$p_y(A - R) = \frac{p_y 40}{p_y} (p_x - p'_x)$$  

$$p_y(A - R) = 40 (p_x - p'_x)$$  

The cost to the government for Plan B is $m' - m$ and now we show that $m' - m = p_y(B - R)$. 

$$p_y(B - R) = p_y \left( \frac{m'}{p_y} - \frac{p_x}{p_y} 40 - \frac{m}{p_y} + \frac{p_x}{p_y} 40 \right)$$  

$$p_y(B - R) = p_y \left( \frac{m'}{p_y} - \frac{m}{p_y} \right)$$  

$$p_y(B - R) = \frac{p_y}{p_y} (m' - m)$$  

$$p_y(B - R) = m' - m$$

(d) Which provides the low income consumer with the highest level of utility: Plan A or Plan B?

**Solution:** Plan B. Not just because of the way it’s drawn, but in general this result will hold. With Plan B, consumers receive extra income they can spend on either day care or other goods. With the subsidy, Plan A, there is an income effect and a substitution effect. The substitution effect results in consumers consuming more day care while keeping their utility fixed. With Plan B, there’s no substitution effect and the utility level is higher.
(e) Which costs the government more: Plan A or Plan C?

Solution:

The cost to the government for Plan A is \((p_x - p'_x)40\) and in relation to the graph above \((p_x - p'_x)40 = p_y(A - R)\). The cost to the government for Plan C is \(40p_x\) since 40 hours of daycare are purchased by the government. In relation to the graph, the cost of Plan C is \(p_y(C - R)\) and now let’s show that \(40p_x = p_y(C - R)\). Note that point C on the graph is the vertical intercept and \(C = \frac{m}{p_y}\).

\[
\begin{align*}
  p_y(C - R) &= p_y \left( \frac{m}{p_y} - \frac{m}{p_y} + \frac{px}{40} \right) \\
  p_y(C - R) &= p_y \left( \frac{px}{40} \right) \\
  p_y(C - R) &= px40
\end{align*}
\]

Plan C costs more.
(f) Compare Plan B to Plan C on the basis of cost to the government. (Pay attention to the income elasticity of day care consumption.)

**Solution:** Either Plan B or Plan C could be more expensive. If the income elasticity of the consumer is sufficiently small, Plan C may be less expensive. You should see this with the Jones family below. Here’s a figure with Plan B and Plan C included. Note that with different preferences, we could draw this figure in a different way to get a different result regarding the cost of Plan B and Plan C.
Now that you’ve done this graphically, lets connect question 4 with some numbers. There are two families: the Smith family and the Jones family. Lets see how the policies from question 4 (Plans A-C) would affect them. Daycare costs $10 per hour \((P_x = 10)\). Both have an initial income of $100 \((m = 100)\). With Plan C, the government provides free child care for 40 hours, and since day care is $10 an hour, Plan C will cost the government $400.

(a) The Smith family has a demand function for daycare given by

\[ x = 10mP_x^{-2} \]

where \(x\) is the number of hours of daycare and \(P_x\) is the price of \(x\).

i. What is their income elasticity of demand?

**Solution:** The income elasticity of demand is the percentage change in quantity demanded divided by the percentage change in income. \(x\) is the quantity demanded and \(m\) is income. The income elasticity of demand is

\[ \frac{\partial x}{\partial m} = \frac{\partial x}{\partial m} \frac{m}{x}. \]

The demand function is

\[ x = 10mP_x^{-2} \]

so

\[ \frac{\partial x}{\partial m} = 10P_x^{-2}. \]

The income elasticity of demand is

\[ \frac{\partial x}{\partial m} \frac{m}{x} = 10P_x^{-2} \frac{m}{10mP_x^{-2}} = 1. \]

ii. Is day care a normal good or an inferior good?

**Solution:** Day care is a normal good since the income elasticity of demand is positive.

iii. What is their own price elasticity of demand?

**Solution:** The own price elasticity of demand is the percentage change in quantity demanded divided by the percentage change in the price. \(x\) is the quantity demanded and \(P_x\) is the price. The own price elasticity of demand is

\[ \frac{\partial x}{\partial P_x} \frac{P_x}{P_x} = \frac{\partial x}{\partial P_x} \frac{P_x}{P_x}. \]
iv. Is demand elastic or inelastic?

**Solution:** Demand is elastic since the own price elasticity of demand is less than -1.

v. How many hours of daycare do they buy before the government intervention?

**Solution:** The demand for day care is

\[ x = 10mP_x^{-2} \]

and the problem gives us \( P_x = 10 \) and \( m = 100 \). The hours of day care purchased before government interaction are

\[ x = 10(100)(10)^{-2} = 10. \]

(b) Under Plan A the government subsidizes child care. What would the subsidy have to be to induce the Smith family to buy 40 hours of daycare?

**Solution:** With \( m = 100 \), figure out the value of \( P_x \) such that \( x = 40 \).

\[
\begin{align*}
x &= 10mP_x^{-2} \\
40 &= 10(100)P_x^{-2} \\
P_x^2 &= \frac{100}{4} \\
P_x &= 5
\end{align*}
\]

If the price were \( P_x = 5 \), then they would buy 40 hours of day care. Since the current price is $10, the subsidy would have to be \( 10 - 5 = 5 \).
(c) How much would that cost the government? (Take the subsidy and multiply it by 40)

Solution: The subsidy would cost $5 \times 40 = 200$.

(d) Under Plan B, the government gives the Smith family enough income to induce them to purchase 40 hours of childcare. How much income would they need to voluntarily buy 40 hours of child care?

Solution: With $P_x = 10$, figure out the value of $m$ such that $x = 40$.

\[
x = 10mP_x^{-2}
\]
\[
40 = 10m(10)^{-2}
\]
\[
m = 400
\]

With $m = 400$, 40 hours of day care would be purchased. Since they have $100, they would need $300 more.

(e) The Jones family has a demand function given by

\[
x = 10m^{\frac{1}{2}}P_x^{-1}
\]

i. What is their income elasticity of demand?

Solution: The income elasticity of demand is the percentage change in quantity demanded divided by the percentage change in income. $x$ is the quantity demanded and $m$ is income. The income elasticity of demand is

\[
\frac{\partial x}{\partial m} \cdot \frac{m}{x} = \frac{\partial x}{\partial m} \frac{m}{x}.
\]

The demand function is

\[
x = 10m^{\frac{1}{2}}P_x^{-1}
\]

so

\[
\frac{\partial x}{\partial m} = 5m^{-\frac{1}{2}}P_x^{-1}.
\]

The income elasticity of demand is

\[
\frac{\partial x}{\partial m} \cdot \frac{m}{x} = 5m^{-\frac{1}{2}}P_x^{-1} \cdot \frac{m}{10m^{\frac{1}{2}}P_x^{-1}} = \frac{1}{2}.
\]
ii. What is their own price elasticity of demand?

Solution: The own price elasticity of demand is the percentage change in quantity demanded divided by the percentage change in the price. \( x \) is the quantity demanded and \( P_x \) is the price. The own price elasticity of demand is

\[
\frac{\partial x}{\partial P_x} = \frac{\partial x}{P_x} \frac{P_x}{x}.
\]

The demand function is

\[
x = 10m^{\frac{1}{2}}P_x^{-1}
\]

so

\[
\frac{\partial x}{\partial P_x} = -10m^{\frac{1}{2}}P_x^{-2}.
\]

The own price elasticity of demand is

\[
\frac{\partial x}{\partial P_x} \frac{P_x}{x} = -10m^{\frac{1}{2}}P_x^{-2} \frac{P_x}{10m^{\frac{1}{2}}P_x^{-1}} = -1.
\]

iii. Which family's demand is more responsive to changes in the price? To changes in income?

Solution: The Smith family is more responsive to both changes in the price and changes in income. The magnitudes of the price elasticity of demand and the income elasticity of demand are greater for the Smith family.

iv. How many hours of daycare do the Jones' buy before the government intervention?

Solution: The demand for day care is

\[
x = 10m^{\frac{1}{2}}P_x^{-1}
\]

and the problem gives us \( P_x = 10 \) and \( m = 100 \). The hours of day care purchased before government interaction are

\[
x = 10(100)^{\frac{1}{2}}(10)^{-1} = 10.
\]
(f) Under Plan A what would the subsidy have to be to induce the Jones family to buy 40 hours of child care?

**Solution:** With \( m = 100 \), figure out the value of \( P_x \) such that \( x = 40 \).

\[
\begin{align*}
x &= 10m^{\frac{1}{2}}P_x^{-1} \\
40 &= 10(100)^{\frac{1}{2}}P_x^{-1} \\
P_x &= \frac{100}{40} \\
P_x &= 2.5
\end{align*}
\]

If the price were \( P_x = 2.5 \), then they would buy 40 hours of day care. Since the current price is $10, the subsidy would have to be \( 10 - 2.5 = 7.5 \).

(g) How much would that cost the government?

**Solution:** The subsidy would cost \( 7.5 \times 40 = 300 \).

(h) Under Plan B, how much income would they need to voluntarily buy 40 hours of child care?

**Solution:** With \( P_x = 10 \), figure out the value of \( m \) such that \( x = 40 \).

\[
\begin{align*}
x &= 10m^{\frac{1}{2}}P_x^{-1} \\
40 &= 10m^{\frac{1}{2}}(10)^{-1} \\
m^{\frac{1}{2}} &= 40 \\
m &= 1600
\end{align*}
\]

With \( m = 1600 \), 40 hours of day care would be purchased. Since they have $100, they would need $1500 more.

Can you see why you got the answers that you did? (Now go back and look at your answers to the last question.)