Instructions:
You must answer all of the following questions. Each question is worth the same amount. You have the class period to complete the exam.

Answer each question clearly and concisely. You must show your work to receive credit.

This exam is given under the rules of the Montana State University. By printing your name above you acknowledge the University’s Honor Code and agree to comply with the provisions of the Honor Code. You may not use notes or receive any assistance. There is to be no talking during the exam. You may use a calculator, but are never allowed to use device allowing you to take photographs or transmit over a network. No notes, no assistance, no talking, no cell phones, but you can use a calculator.

Clearly print your name above, in the space provided on the next page and in your blue book(s). You must turn in your blue book(s).
1. Consider a linear city Hotelling model. There are two firms, A and B, located at the ends of the product space. The length of the product space is 3 and transportation costs are 1 times the distance traveled. Each consumer has a baseline valuation of 9 and each firm has a constant marginal cost of 4. Answer the following questions for the competitive equilibrium if it exists.

(a) What is each firm’s best response function?

**Solution:** Start with the consumer’s utility maximization problem. \( u_A(x) = 9 - p_A - x \) and \( u_B(x) = 9 - p_B - (3 - x) \). A consumer located at \( x \) in the preference space buys from whatever firm provides greater utility.

Now think about the indifferent consumer(s). For a competitive equilibrium, the indifferent consumer is located at \( x \) and is indifferent to buying from A or B where

\[
\begin{align*}
\frac{d}{dx} (u_A(x) - u_B(x)) &= 0 \\
9 - p_A - x &= 9 - p_B - (3 - x) \\
2x &= p_B - p_A + 3 \\
x &= \frac{(p_B - p_A)}{2} + \frac{3}{2}
\end{align*}
\]

With the indifferent consumers we define the demand for each firm. Each firm’s demand is defined piecewise based on whether their in a competitive equilibrium or have a local monopoly, but we’ll only focus on the case of a competitive equilibrium.

\[
q_A = \frac{(p_B - p_A)}{2} + \frac{3}{2} \\
q_B = 3 - \left( \frac{(p_B - p_A)}{2} + \frac{3}{2} \right)
\]

Each firm’s profits are \( \pi_i = p_iq_i - 4q_i = (p_i - 4)q_i \) for \( i = \{A, B\} \) and after substitution and rearranging we get the following.

\[
\begin{align*}
\pi_A &= (p_A - 4) \left( \frac{(p_B - p_A)}{2} + \frac{3}{2} \right) \\
\pi_B &= (p_B - 4) \left( \frac{(p_A - p_B)}{2} + \frac{3}{2} \right)
\end{align*}
\]

Each firm chooses their price to maximize profits. The first order conditions are as follows.

\[
\begin{align*}
\frac{d\pi_A}{dp_A} &= \left( \frac{(p_B - p_A)}{2} + \frac{3}{2} \right) - \left( \frac{(p_A - 4)}{2} \right) \\
\frac{d\pi_B}{dp_B} &= \left( \frac{(p_A - p_B)}{2} + \frac{3}{2} \right) - \left( \frac{(p_B - 4)}{2} \right)
\end{align*}
\]
Setting $\frac{d\pi_i}{dp_i} = 0$ and solving for that price we get the following. For firm A:

$$0 = \left(\frac{(p_B - p_A)}{2} + \frac{3}{2}\right) - \left(\frac{(p_A - 4)}{2}\right)$$

$$p_A = \frac{p_B}{2} + \frac{7}{2}$$

For firm B:

$$0 = \left(\frac{(p_A - p_B)}{2} + \frac{3}{2}\right) - \left(\frac{(p_B - 4)}{2}\right)$$

$$p_B = \frac{p_A}{2} + \frac{7}{2}$$

These are the firms best response functions.

$$p_A = \frac{p_B}{2} + \frac{7}{2}$$

$$p_B = \frac{p_A}{2} + \frac{7}{2}$$

(b) Are the two goods strategic complements or strategic substitutes?

**Solution:** The two goods are strategic complements.

(c) What are the equilibrium prices and quantities?

**Solution:** The equilibrium will be a symmetric equilibrium. Imposing the symmetric equilibrium conditions of $p_A = p_B = p^*$ we get the following.

$$p^* = \frac{p^*}{2} + \frac{7}{2}$$

$$q^* = \frac{3}{2}$$

Solving for $p^*$ and substituting in $p^*$ to the above equations yields the following.

$$p^* = 7$$

$$q^* = \frac{3}{2}$$

For a competitive equilibrium it must be that $p^* = 7$ and $q^* = \frac{3}{2}$. 
(d) Find the profits of each firm.

**Solution:** Each firm’s profits are \( (p^* - 4) \left( \frac{3}{2} \right) = \frac{9}{2} \).

(e) Find the consumer surplus.

**Solution:**

\[
CS = 2 \int_0^{\frac{3}{2}} (9 - 7 - x) \, dx = 2 \left[ 2x - \frac{x^2}{2} \right]_0^{\frac{3}{2}} = \frac{15}{4}
\]

2. Consider a Salop circular model of product differentiation with equally spaced firms. The circumference of the circle is 10. Each consumer has transportation costs of 1 times the distance traveled and a baseline valuation of 100. Firms have constant marginal costs of 5. There are \( N \) firms. Answer the following question for a competitive equilibrium assuming that it exists.

(a) Express a firm’s demand as a function of the neighboring firm’s prices and the number of firms. What will a firm’s demand be in equilibrium?

**Solution:** Let Firm \( A \) be located at zero. The length between each firm is \( 10/N \). The indifferent consumer between Firm \( A \) and Firm \( B \) to the right is as follows.

\[
100 - p_A - x = 100 - p_B - \frac{10}{N} + x
\]

\[
x = \frac{p_B - p_A}{2} + \frac{5}{N}
\]

The quantity demanded from the consumers to the right of Firm \( A \) is \( \frac{p_B - p_A}{2} + \frac{5}{N} \).

Likewise the quantity demanded from the consumers to the left of Firm \( A \) (with Firm \( C \) being the closest to Firm \( A \)) is \( \frac{p_C - p_A}{2} + \frac{5}{N} \).

The firms demand is \( \frac{p_B - p_A}{2} + \frac{p_C - p_A}{2} + \frac{10}{N} \). In equilibrium, all the prices are the same \( (p_A = p_B = p_C = \ldots = p^*) \) and equilibrium demand is \( \frac{10}{N} \).
(b) Setup a firm’s profit maximization problem and determine a firm’s best response as a function of the number of firms and neighboring firm’s prices.

Solution:

\[ \max_{p_A} \pi = (p_A - 5) \left( \frac{p_B - p_A}{2} + \frac{p_C - p_A}{2} + \frac{10}{N} \right) \]

\[ \frac{\partial \pi}{\partial p_A} = \frac{p_B - p_A}{2} + \frac{p_C - p_A}{2} + \frac{10}{N} - p_A + 5 = 0 \]

\[ p_B + p_C - 2p_A + \frac{20}{N} - 2p_A + 10 = 0 \]

\[ 4p_A = p_B + p_C + \frac{20}{N} + 10 \]

\[ p_A = \frac{p_B + p_C}{4} + \frac{5}{N} + \frac{5}{2} \]

(c) What is the equilibrium price as a function of the number of firms?

Solution: Impose price symmetry on the best response found above.

\[ p_A = \frac{p_B + p_C}{4} + \frac{5}{N} + \frac{5}{2} \]

\[ p^* = \frac{p^*_B}{2} + \frac{5}{N} + \frac{5}{2} \]

\[ \frac{p^*_B}{2} = \frac{5}{N} + \frac{5}{2} \]

\[ p^* = \frac{10}{N} + 5 \]

(d) Determine the following comparative static: \( \frac{\partial \pi}{\partial N} \).

Solution: From above, \( \pi = (p_A - 5) \left( \frac{10}{N} \right) \), and \( p_A - 5 = \frac{10}{N} \). The equilibrium profits are \( \pi = \left( \frac{10}{N} \right)^2 \). \( \frac{\partial \pi}{\partial N} = -2\frac{10^2}{N^2} < 0 \).
3. The inverse demand function is \( P = 120 - 2Q \). Two firms compete in quantities and each firm has a cost of \( C(q) = 2q^2 \). The interest rate is 25%.

(a) If a competitive equilibrium is maintained over time, what is the discounted sum of profits?

**Solution:** Firm A’s problem is as follows.

\[
\max_{q_A} \pi_A = (120 - 2q_A - 2q_B)q_A - 2q_A^2
\]

The first order condition is

\[
\frac{d\pi_A}{dq_A} = 120 - 2q_B - 8q_A
\]

and setting this equal to zero and solving for \( q_A \) yields the following.

\[
q_A = 15 - \frac{q_B}{4}
\]

The above is a firm’s best response which will be used later.

Imposing symmetry conditions and solving for the equilibrium quantity yields \( q^* = 12 \), \( Q^* = 24 \), and \( P^* = 72 \). The per period competitive profits are

\[
\pi^C = 72(12) - 2(12)^2 = 576
\]

The discounted sum of profits are

\[
\frac{1}{r} \pi^C = 4(576) = 2304.
\]

(b) If firms can perfectly collude and share the profits evenly, what would be the per period profits of each firm?

**Solution:** For a collusive equilibrium, \( Q = 2q \) and a firm’s profit is

\[
\max_q \pi = (120 - 2(2q))q - 2q^2
\]

The first order condition is

\[
\frac{d\pi}{dq} = 120 - 12q
\]

and setting this equal to zero and solving for \( q \) yields the following.

\[
q^* = 10
\]

\[
Q^* = 20
\]

\[
P^* = 120 - 2(20) = 80
\]
The per period collusive profits are

\[ \pi^{TC} = 80(10) - 2(10)^2 = 600 \]

(c) Consider a trigger strategy where if a firm deviates from a collusive equilibrium, the other firm behaves competitively forever after. What is the discounted sum of profits obtained by deviating from a collusive equilibrium and is it possible to maintain a collusive equilibrium if the other firm plays the trigger strategy?

**Solution:** Assume firm A deviates from the collusive equilibrium. We first need to find firm A’s optimal deviation. In a collusive equilibrium, \( q^* = 10 \), so assume \( q_B = 10 \) and firm A plays their best response. Firm A’s best response is \( q_A = 15 - \frac{q_B}{4} = \frac{25}{2} \). The market quantity is then \( Q = \frac{45}{2} \) and the price is \( P = 120 - 2 \frac{25}{2} = 75 \). Firm A’s profits from the deviation are

\[ \pi^D_A = 75 \left( \frac{25}{2} \right) - 2 \left( \frac{25}{2} \right)^2 = 625 \]

The discounted sum of profits would then be as follows.

\[ \frac{1}{1 + r} \pi^D_A + \frac{1}{1 + r} \pi^C + \frac{4}{5}(625) + \frac{4}{5}(4)(576) = \frac{11716}{5} \]

It’s possible to maintain a collusive equilibrium if the following condition holds.

\[ 2400 > \frac{11716}{5} \]

which is does, so yes it is possible to maintain a collusive equilibrium if the other firm plays the trigger strategy.

(d) Consider another type of trigger strategy where one firm tries to establish a collusive equilibrium from the competitive equilibrium. If one firm deviates from the competitive quantity to the collusive quantity, then the other firm will play along forever after (unless one firm deviates from the collusive equilibrium). What is the discounted sum of profits obtained by deviating from a competitive equilibrium and is it possible to switch from a competitive equilibrium to a collusive one?

**Solution:** Assume firm A deviates from the competitive equilibrium. From the question, we know that firm B plays the competitive outcome of \( q_B = 12 \) and firm A plays the collusive outcome of \( q_A = 10 \). The market quantity is then \( Q = 22 \) and the price is \( P = 120 - 2(22) = 76 \). Firm A’s profits from the
deviation are
\[ \pi_A^D = 76 (10) - 2 (10)^2 = 560 \]

Note that this is not a profitable deviation for firm A, but if they can then get firm B to play along, the discounted sum of profits for firm A is as follows.
\[ \frac{1}{1 + r} \pi_A^D + \frac{1}{1 + r} \pi_{TC} \]
\[ \frac{4}{5} (560) + \frac{4}{5} (4)(600) = 2368 \]

It's possible to switch from a competitive equilibrium to a collusive one if 2368 > 2304, so yes it is possible.

4. The inverse market demand is \( P = 120 - 2Q \), costs for Firm \( i \) are \( C(q_i) = 20q_i \) and there are four firms. Assume firms compete in quantities.

(a) What is the competitive equilibrium price and the profit of each firm?

\[ \text{Solution:} \] Setup Firm A's problem and then solve for the symmetric equilibrium. Firm A's quantity is \( q_A \) and the quantity of all other firms other than Firm A is \( q_{-A} \).

\[
\begin{align*}
\max_{q_A} \pi_A &= (120 - 2q_A - 2q_{-A})q_A - 20q_A \\
\frac{\partial \pi_A}{\partial q_A} &= 120 - 4q_A - 2q_{-A} - 20 = 0 \\
100 - 2q_{-A} &= 4q_A \\
100 - 2(3q^*) &= 4q^* \\
q^* &= 10 \\
Q^* &= 40 \\
P^* &= 120 - 2(40) = 40 \\
\pi_i^* &= 40(10) - 20(10) = 200
\end{align*}
\]

(b) If all firms perfectly collude, what is the collusive equilibrium price and the profit of each firm?

\[ \text{Solution:} \] If all firms perfectly collude, assume that \( Q = 4q \) so that they split
the market evenly. One firm’s problem is as follows:

\[
\max_q \pi = (120 - 2(4q))q - 20q
\]

\[
\frac{\partial \pi}{\partial q} = 120 - 16q - 20 = 0
\]

16q = 100

\[q^* = \frac{25}{4}\]

\[Q^* = 25\]

\[P^* = 120 - 2(25) = 70\]

\[\pi_i^* = 70 \left( \frac{25}{4} \right) - 20 \left( \frac{25}{4} \right) = 312.5\]

(c) Assume the four firms perfectly collude. A policy maker asks you to come up with a reason why they should do something about collusion and support your explanation with numerical evidence from this problem. How do you respond?

**Solution:** You could argue that collusion reduces consumer surplus. In the absence of collusion

\[CS = \frac{1}{2}(120 - 40)40 = 1600\]

and with collusion

\[CS = \frac{1}{2}(120 - 70)25 = 625\]

so the change in consumer surplus is

\[\Delta CS = -975.\]

Collusion makes consumers worse off by $975.

You could also make an argument concerning the total surplus or overall efficiency. In the absence of collusion, industry profits are $800 and the overall efficiency is $2,400. With collusion, industry profits are $1,250 and the overall efficiency is $1,875. Collusion makes the firms better off, the consumers worse off and leads to a lower total surplus.

(d) Another policy maker argues that all policy makers have perfect information/foresight and that they have the ability to construct optimal transfers. Why might policy makers still want to do something about collusion and support your explanation with numerical evidence from this problem?
Solution: Transfers would allow you to take the gains of the producers and distribute them to the consumers. Since total surplus decreases by $525, the gains of the producers don’t cover the loss of the consumers. Overall efficiency would increase if policy makers did something about collusion.