Instructions:
There are 16 questions worth a total of 100 points. Answer each question clearly and concisely. You must show your work to receive credit. You are allowed to work with others, but all work must be your own.

Clearly print your name above and in the space provided on the next page. You must turn in both sides of this cover sheet along with your responses. You do not need to turn in the questions, only your responses with the cover sheet. All pages must be stapled to be graded.
Math for Math’s Sake

The following is meant to be a review of the math you’re supposed to know but might need some refreshing.

1. Find $f’(x)$ or $\frac{df(x)}{dx}$ for the following functions:

(a) $f(x) = Ax^n$ where $n$ and $A$ are constants

Solution: $f’(x) = nAx^{n-1}$

(b) $f(x) = x^{0.5}$

Solution: $f’(x) = 0.5x^{-0.5} = \frac{0.5}{\sqrt{x}}$

(c) $f(x) = 10x^{-3}$

Solution: $f’(x) = -30x^{-4} = -\frac{30}{x^4}$

(d) $f(x) = 5x^4 + 6x^2 + 7x + 3 + \frac{22}{x}$

Solution: $f’(x) = 20x^3 + 12x + 7 - \frac{22}{x^2}$

(e) $f(x) = g(x)h(x)$

Solution: $f’(x) = g’(x)h(x) + g(x)h’(x)$ (The product rule)

(f) $f(x) = (x^2)(x^3 - 4)$

Solution: $f’(x) = (2x)(x^3 - 4) + (x^2)(3x^2) = 5x^4 - 8x$

(g) $f(x) = (3x^3)(\ln(x))$

Solution: $f’(x) = 9x^2 \ln(x) + 3x^3 \left(\frac{1}{x}\right)$

(h) $f(x) = \frac{g(x)}{h(x)}$

Solution: $f’(x) = \frac{g’(x)h(x) - g(x)h’(x)}{(h(x))^2}$ (The quotient rule)

(i) $f(x) = \frac{\ln(x)}{5x^2}$

Solution: $f’(x) = \frac{5x^2 \ln(x) - 10x}{25x^4} = \frac{5x - \ln(x)10x}{25x^4} = \frac{1-2\ln(x)x^3}{5x^3}$
(j) \( f(x) = g(h(x)) \)

**Solution:** \( f'(x) = g'(h(x))h'(x) \) (The chain rule)

(k) \( f(x) = (x^5 - 10x)^a \) where \( a \) is a constant

**Solution:** \( f'(x) = a(x^5 - 10x)^{a-1}(5x^4 - 10) \)

(l) \( f(x) = \ln(2x) \)

**Solution:** \( f'(x) = \frac{1}{2x}(2) = \frac{1}{x} \)

2. In question 1.c above, what is the slope of \( f(x) \) when \( x = 2 \)?

**Solution:** From question 1.c, \( f'(x) = -30x^{-4} \) and \( f'(2) = -30(2)^{-4} = -\frac{15}{8} = -1.875 \)

3. Find all the local extrema for the following functions and indicate whether the point is a local maximum or a local minimum.

(a) \( f(x) = 5x^2 + 30x + 3 \)

**Solution:** \( f'(x) = 10x + 30 \) and \( f''(x) = 10 > 0 \Rightarrow \text{Convex function (min)} \)

\( f'(x) = 0 \Rightarrow 10x + 30 = 0 \Rightarrow x = -3 \) is a local minimizer

(b) \( f(x) = -8x^2 + 4x - 50 \)

**Solution:** \( f'(x) = -16x + 4 \) and \( f''(x) = -16 < 0 \Rightarrow \text{Concave function (max)} \)

\( f'(x) = 0 \Rightarrow -16x + 4 = 0 \Rightarrow x = \frac{1}{4} \) is a local maximizer

4. Find the inverse of the following functions:

(a) \( y = f(x) \rightarrow y = x^2 + 4 \) with \( x \geq 0 \)

**Solution:** \( x = \sqrt{y - 4} \)

(b) \( p = f(Q) \rightarrow p = 3 - 6Q \)

**Solution:** \( Q = \frac{3-p}{6} \)
(c) \( p = f(Q) \rightarrow p = a - bQ \)

Solution: \( Q = \frac{a}{b} - \frac{p}{b} \)

5. Let \( f(x) \) be defined as follows where \( x \in [0, 5] \) means \( 0 \leq x \leq 5 \):
\[
f(x) = \begin{cases} 
-2x + 15 & \text{for } x \in [0, 5], \\
-4x + 25 & \text{for } x \in [5, 6].
\end{cases}
\]

(a) What is the slope of \( f(x) \) when \( x = 4 \)?

Solution: \(-2\)

(b) What is the slope of \( f(x) \) when \( x = 5.5 \)?

Solution: \(-4\)

(c) What is the slope of \( f(x) \) when \( x = 5 \)?

Solution: The slope is undefined when \( x = 5 \). For a derivative to exist, the function must be continuous and smooth. The above function is continuous, but has a kink so it is not smooth (The limit from the right does not equal the limit from the left.).

6. Solve the following integrals:

(a) \( \int_0^5 2xdx \)

Solution: \( \int_0^5 2xdx = [x^2]\big|_0^5 = 25 \)

(b) \( \int_1^4 \frac{1}{x}dx \)

Solution: \( \int_1^4 \frac{1}{x}dx = [\ln(x)]_1^4 = \ln(4) \)

7. Simplify the following:
\[
\frac{x^{\frac{1}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}
\]

Solution:
\[
\frac{x^{\frac{1}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}} = \frac{x^{\frac{1}{3}}x^{\frac{2}{3}}}{y^{\frac{2}{3}}y^{\frac{1}{3}}} = \frac{x^{\frac{1}{3} + \frac{2}{3}}}{y^{\frac{2}{3} + \frac{1}{3}}} = \frac{x}{y}
\]
8. Solve the following for $x$: \( \frac{x}{2} = \frac{8}{x^3} \)

\[ \text{Solution:} \quad \text{cross multiply to get } x^4 = 16 \text{ so } x = \pm 2 \]

9. Solve the system of two equations and two unknowns and graph the two equations:

\[
\begin{align*}
Q &= 2p - 40 \\
Q &= 200 - p.
\end{align*}
\]

\[ \text{Solution:} \]

\[
\begin{align*}
2p - 40 &= 200 - p \\
3p &= 240 \\
p &= 80 \\
Q &= 200 - (80) \text{ or } Q = 2(80) - 40 \\
Q &= 120
\end{align*}
\]
10. Solve the system of two equations and two unknowns:

\[ q_1 + 3q_2 = 10 \]
\[ 4q_1 + 8q_2 = 4. \]

**Solution:**

\[ q_1 = 10 - 3q_2 \]

Take the first equation and solve for \( q_1 \).

\[ 4(10 - 3q_2) + 8q_2 = 4 \]

Substitute \( q_1 \) into the second equation.

\[ 40 - 12q_2 + 8q_2 = 4 \]

Simplify

\[ 36 = 4q_2 \]

Simplify

\[ q_2 = 9 \]

Only part of the solution

\[ q_1 = 10 - 3(9) = -17 \]

Substitute \( q_2 = 9 \) into the equation for \( q_1 \).

11. Take the partial derivative of the following with respect to \( x_1 \) and \( x_2 \) where \( a, b \) and \( c \) are constants:

(a) \( U = 2x_1 + 3x_2 \)

**Solution:**

\[ \frac{\partial U}{\partial x_1} = 2 \quad \frac{\partial U}{\partial x_2} = 3 \]

(b) \( U = x_1^{\frac{1}{2}} + x_2 \)

**Solution:**

\[ \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} \quad \frac{\partial U}{\partial x_2} = 1 \]

(c) \( U = x_1 + \frac{1}{x_2} \)

**Solution:**

\[ \frac{\partial U}{\partial x_1} = 1 \quad \frac{\partial U}{\partial x_2} = -\frac{1}{x_2^2} \]

(d) \( U = x_1^{a} + x_2^{b} \)

**Solution:**

\[ \frac{\partial U}{\partial x_1} = ax_1^{a-1} \quad \frac{\partial U}{\partial x_2} = bx_2^{b-1} \]

(e) \( U = x_1^{0.3} + x_2^{0.7} \)

**Solution:**

\[ \frac{\partial U}{\partial x_1} = 0.3x_1^{-0.7} \quad \frac{\partial U}{\partial x_2} = 0.7x_2^{-0.3} \]
(f) \( U = x_1^2 x_2^3 \)

Solution: \( \frac{\partial U}{\partial x_1} = 2x_1 x_2^3 \quad \frac{\partial U}{\partial x_2} = 3x_1^2 x_2^2 \)

(g) \( U = c x_1^a x_2^b \)

Solution: \( \frac{\partial U}{\partial x_1} = acx_1^{a-1} x_2^b \quad \frac{\partial U}{\partial x_2} = bcx_1^a x_2^{b-1} \)

**Cost Analysis**

These questions review the basic properties of cost functions using calculus.

12. Given a linear cost curve of \( C(Q) = 5Q + 8 \):

(a) Find the Fixed Cost (FC).

Solution: \( FC = 8 \)

(b) Find the Variable Cost (VC).

Solution: \( VC = 5Q \)

(c) Find the Average Cost (AC).

Solution: \( AC = 5 + \frac{8}{Q} \)

(d) Find the Average Fixed Cost (AFC).

Solution: \( AFC = \frac{8}{Q} \)

(e) Find the Average Variable Cost (AVC).

Solution: \( AVC = 5 \)

(f) Find the Marginal Cost (MC).

Solution: \( MC = 5 \)
(g) Plot the cost curve with the FC and VC curves on the same plot.

![Diagram showing cost curves with FC and VC](image-url)
(h) Plot the AC, AFC, AVC and MC curves on the same plot.

13. Given a quadratic cost curve of $C(Q) = 2Q^2 + 5Q + 8$:
   
   (a) Find the Fixed Cost (FC).
   
   **Solution:** $FC = 8$

   (b) Find the Variable Cost (VC).
   
   **Solution:** $VC = 2Q^2 + 5Q$

   (c) Find the Average Cost (AC).
   
   **Solution:** $AC = 2Q + 5 + \frac{8}{Q}$

   (d) Find the Average Fixed Cost (AFC).
   
   **Solution:** $AFC = \frac{8}{Q}$
(e) Find the Average Variable Cost (AVC).

**Solution:** $AVC = 2Q + 5$

(f) Find the Marginal Cost (MC).

**Solution:** $MC = 4Q + 5$

(g) What is the minimum of average cost and at what quantity does this occur?

**Solution:**

Differentiate $AC(Q)$ to get $AC'(Q) = 2 - \frac{8}{Q^2}$

Differentiate $AC''(Q)$ to get $AC''(Q) = \frac{16}{Q^3} > 0$

$AC$ is a convex function (finding a min)

Set $AC''(Q) = 0$ to find the minimum $AC$

$0 = 2 - \frac{8}{Q^2}$

Solve for $Q$ to get $Q = 2$

Minimum of average cost is $AC(Q = 2)\quad AC(Q = 2) = 2(2) + 5 + \frac{8}{2} = 13$

(h) Show that marginal cost equals average cost at the minimum of average cost.

**Solution:**

Set marginal cost equal to average cost $4Q + 5 = 2Q + 5 + \frac{8}{Q}$

Simplify $2Q = \frac{8}{Q}$

Solve for $Q\quad Q = 2$

$MC = AC$ at $Q = 2$ and the minimum of $AC$ occurs when $Q = 2$
Revenue Analysis

These questions review the basic connections between demand, revenue and the price elasticity of demand using calculus. Denote the price elasticity of demand as $\xi$, and for those of you who don’t remember:

$$\xi = \frac{\text{Percentage change in quantity demanded}}{\text{Percentage change in price}}$$

$$\xi = \frac{\% \Delta \text{ in } Q}{\% \Delta \text{ in } p}$$

$$\xi = \frac{\left(\frac{dQ}{Q}\right)}{\left(\frac{dp}{p}\right)}$$

$$\xi = \left(\frac{dQ}{dp}\right) \left(\frac{p}{Q}\right)$$

14. Starting with a generic linear inverse demand function $p = a - bQ$ where $a$ and $b$ are parameters:
   (a) Find the demand function, $Q(p)$ (the inverse of the inverse demand)

   **Solution:** Rearranging $p = a - bQ$ gives $Q = \frac{a}{b} - \frac{p}{b}$

   (b) Find $\frac{dQ}{dp}$

   **Solution:** Since $Q(p) = \frac{a}{b} - \frac{p}{b}$, $\frac{dQ}{dp} = -\frac{1}{b}$

   (c) Find $\xi$ as a function of $Q$ and the parameters $a$ and $b$ (not $p$)

   **Solution:**

   $$\xi = \left(\frac{dQ}{dp}\right) \left(\frac{p}{Q}\right)$$

   from above

   $$\xi = \left(-\frac{1}{b}\right) \left(\frac{p}{Q}\right)$$

   from part b

   $$\xi = \left(-\frac{1}{b}\right) \left(\frac{a - bQ}{Q}\right)$$

   substitute in the inverse demand

   $$\xi = -\frac{a}{bQ} + 1$$

   by simplification
15. Given an inverse demand function of \( p = 20 - 4Q \):  
   (a) Find the revenue function \( R(Q) \).  

   **Solution:** Revenue is price times quantity, \( R(Q) = pQ = (20 - 4Q)Q = 20Q - 4Q^2 \)

   (b) Find the marginal revenue function \( MR(Q) \).  

   **Solution:** Marginal revenue is \( R'(Q) \), so \( MR(Q) = 20 - 8Q \)

   (c) Find the price elasticity of demand \( \xi(Q) \).  

   **Solution:** From Question 14.c, \( \xi = -\frac{a}{bQ} + 1 \). In this case \( a = 20 \) and \( b = 4 \), so \( \xi(Q) = -\frac{5}{Q} + 1 \).

   (d) Show that revenue is maximized when the price elasticity of demand is unit elastic.  

   **Solution:** Note that \( \xi = -1 \) corresponds to unit elasticity. Maximum revenue is determined by setting \( MR(Q) = 0 \Rightarrow 8Q = 20 \Rightarrow Q = 2.5 \) when revenue is maximized \( (MR'(Q) = -8 < 0 \text{ so we're finding a max}) \). Now \( \xi(Q = 2.5) = -\frac{5}{2.5} + 1 = -1 \).

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**Profit Analysis**

Combining the cost analysis and the revenue analysis we get the following questions.

16. Given an inverse demand function of \( p = 20 - 4Q \) find a firm’s maximum profits and the profit maximizing quantity when the cost function is:  
   (a) \( C(Q) = 5Q + 8 \)

   **Solution:** Profits, \( \pi \), are defined as the firm’s revenue, \( R(Q) \) minus its cost, \( C(Q) \). To maximize \( \pi(Q) = R(Q) - C(Q) \), differentiate with respect to the quantity and set \( \pi'(Q) = 0 \). We have \( 0 = R'(Q) - C'(Q) \) and rearranging yields \( R'(Q) = C'(Q) \) which simply states that to maximize profits, set the marginal revenue equal to the marginal cost. When \( p = 20 - 4Q \), \( R(Q) = 20Q - 4Q^2 \) and \( MR(Q) = 20 - 8Q \). The marginal cost is \( MC(Q) = 5 \). Setting \( MR(Q) = MC(Q) \) we get \( 20 - 8Q = 5 \). The profit maximizing quantity is \( Q = \frac{15}{8} \) and the maximum profits are \( \pi \left(Q = \frac{15}{8}\right) = 20 \left(\frac{15}{8}\right) - 4 \left(\frac{15}{8}\right)^2 - 5 \left(\frac{15}{8}\right) - 8 \).
(b) \( C(Q) = 2Q^2 + 5Q + 8 \)

**Solution:** Profits, \( \pi \), are defined as the firm’s revenue, \( R(Q) \) minus its cost, \( C(Q) \). To maximize \( \pi(Q) = R(Q) - C(Q) \), differentiate with respect to the quantity and set \( \pi'(Q) = 0 \). We have \( 0 = R'(Q) - C'(Q) \) and rearranging yields \( R'(Q) = C'(Q) \) which simply states that to maximize profits, set the marginal revenue equal to the marginal cost. When \( p = 20 - 4Q \), \( R(Q) = 20Q - 4Q^2 \) and \( MR(Q) = 20 - 8Q \). The marginal cost is \( MC(Q) = 4Q + 5 \). Setting \( MR(Q) = MC(Q) \) we get \( 20 - 8Q = 4Q + 5 \). The profit maximizing quantity is \( Q = \frac{15}{12} \) and the maximum profits are \( \pi \left( Q = \frac{15}{12} \right) = 20 \left( \frac{15}{12} \right) - 4 \left( \frac{15}{12} \right)^2 - 2 \left( \frac{15}{12} \right)^2 - 5 \left( \frac{15}{12} \right) - 8 \).