Industrial Organization  
ECNS 406  
Fall 2015  

Homework #: 2  

Due by the beginning of class on: Thursday September 10, 2015  

Name:  

Instructions:  
There are 7 questions worth a total of 100 points. Answer each question clearly and concisely. You must show your work to receive credit. You are allowed to work with others, but all work must be your own.  
Clearly print your name above and in the space provided on the next page. You must turn in both sides of this cover sheet along with your responses. You do not need to turn in the questions, only your responses with the cover sheet. All pages must be stapled to be graded.
Game Theory

1. Model the game of Rock-Paper-Scissors as a strategic form game. (10)

(a) Explicitly list the strategic form structure of the game and the elements of the game along with a brief discussion.

Solution:

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>P</td>
</tr>
<tr>
<td>R</td>
<td>(0,0)</td>
<td>(-1,1)</td>
</tr>
<tr>
<td>P</td>
<td>(1,-1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>S</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>

In the figure, there are two players: 1 and 2. The strategy set of each player is to choose one of the following \{ Rock (R), Paper (P), Scissors (S)\}. Each player chooses their strategy simultaneously. Both players have full information about the game, they just don’t know the strategy of the other player. The payoffs of the game are indicated by the strategic form representation in the figure. The payoffs are listed as the (payoff to the player on the left, payoff to the player on top). For instance, if we assume that the players are playing for one dollar, paper beats rock and the winner gets one dollar from the loser.

(b) Derive the best response functions.

Solution: The game is symmetric, so we only really need to derive one best response function. Let’s consider the best response of player $i$ conditional on the strategy that player $j$ uses. Let player $i$’s strategy be labeled as $S_i$ and player $j$’s strategy be labeled as $S_j$.

If player $j$ plays $R$ ($S_j = R$), player $i$ gets:

1. 0 for $S_i = R$
2. 1 for $S_i = P$
3. -1 for $S_i = S$

If player $j$ plays $P$ ($S_j = P$), player $i$ gets:
1. -1 for $S_i = R$
2. 0 for $S_i = P$
3. 1 for $S_i = S$

If player $j$ plays $S$ ($S_j = S$), player $i$ gets:

1. 1 for $S_i = R$
2. -1 for $S_i = P$
3. 0 for $S_i = S$

Player $i$’s best response is a function ($S_i =$) describing what they should do conditional on player $j$’s decision ($S_j$). Player $i$’s best response is

$$S_i = \begin{cases} 
    P & \text{if } S_j = R \\
    S & \text{if } S_j = P \\
    R & \text{if } S_j = S
\end{cases}$$
2. Use the extensive form game in Figure 1 to answer the questions below. The payoffs listed at the bottom are given as (payoff to Player 1, payoff to Player 2), and the ellipse around Player 2’s decision indicates that Player 2 does not know Player 1’s choice.

Figure 1: An extensive form game for Question 2

(a) Rewrite the following extensive form game as a strategic form game.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy</td>
<td>L</td>
</tr>
<tr>
<td>T</td>
<td>(3,2)</td>
</tr>
<tr>
<td>M</td>
<td>(2,4)</td>
</tr>
<tr>
<td>B</td>
<td>(4,3)</td>
</tr>
</tbody>
</table>

Solution:
(b) Derive the best response function for each player.

**Solution:** Let the strategy of player 1 be $S_1$ where $S_1 = L$ or $R$. Let the strategy of player 2 be $S_2$ where $S_2 = T$, $M$, or $B$. The best response functions are as follows.

$$
S_1 = \begin{cases}
R & \text{if } S_2 = T \\
L & \text{if } S_2 = M \\
L \text{ or } R & \text{if } S_2 = B
\end{cases}
$$

$$
S_2 = \begin{cases}
B & \text{if } S_1 = L \\
B & \text{if } S_1 = R
\end{cases}
$$

(c) Are there any Nash Equilibria of the game and if so what are they?

**Solution:** There are two Nash Equilibrium of the game: $(B, L)$ and $(B, R)$. There is no profitable unilateral deviation from either $(B, L)$ and $(B, R)$.

3. Consider the following strategic form game in Figure 2. The payoffs are listed as the (payoff to the player on the left, payoff to the player on top).

**Figure 2: A strategic form game for Question 3**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strategy</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Player 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>(-1,2)</td>
<td>(-1,3)</td>
</tr>
<tr>
<td>B</td>
<td>(-2,-2)</td>
<td>(0,1)</td>
</tr>
</tbody>
</table>
(a) Derive the best response function for each player.

**Solution:** Let the strategy of player 1 be $S_1$ where either $S_1 = L$ or $R$. Let the strategy of player 1 be $S_2$ where either $S_2 = T$ or $B$. The best response for player for each player is as follows.

$$S_1 = \begin{cases} R & \text{if } S_2 = T \\ R & \text{if } S_2 = B \end{cases}$$

$$S_2 = \begin{cases} T & \text{if } S_1 = L \\ B & \text{if } S_1 = R \end{cases}$$

(b) What is the Nash Equilibrium of this game and is it unique?

**Solution:** The unique Nash Equilibrium of the game is $(R, B)$.

(c) How does the outcome change if instead of simultaneous decisions, we model the game with sequential decisions and Player 1 chooses first?

**Solution:** If Player 1 chooses first and Player 2 chooses second, we can use Player 2’s best response. If Player 1 chooses $S_1 = L$, we know that $S_2 = T$ and the payoff to Player 1 is 2. If Player 1 chooses $S_1 = R$, we know that $S_2 = B$ and the payoff to Player 1 is 1. In the sequential game, the outcome is $(L, T)$ which is different from the outcome of the simultaneous game.
4. Types of games:
   (a) Provide an example of a zero sum game.

   **Solution:** See Rock-Paper-Scissors above.

   (b) Provide an example of a coordination or prisoner’s dilemma game.

   **Solution:** See Figure 5 where $A < 5$.

   (c) How do zero sum games differ from coordination or prisoner’s dilemma games?

   **Solution:** In zero sum games, one player’s positive payoffs come at the expense of another player. In a coordination or prisoner’s dilemma game, this is not the case. Usually if decisions can be coordinated, both players can be mutually better off.

   (d) How does repetition change the outcomes of zero sum games and coordination or prisoner’s dilemma games?

   **Solution:** Repeating a coordination or prisoner’s dilemma game, allows a player to punish the other player in future games for failing to coordinate in the present. This increases the benefits of coordination (increases the costs of defection). Repetition of a zero sum game does not change the outcome. The only possibility where repetition of a zero sum game might change the outcome is when the other player consistently plays a strategy that is not a best response. For example, another player might have a bad poker face and you can tell whether or not their bluffing.

   (e) How does adding more players change the outcomes of zero sum games and coordination or prisoner’s dilemma games?

   **Solution:** More players in a zero sum game may either reduce the cost of the losers or may increase the payoff of the winner. More players in a coordination or prisoner’s dilemma game make coordination more difficult and defection more likely.
5. Consider the strategic form game of Figure 3. The payoffs are listed as the (payoff to the player on the left, payoff to the player on top).

(a) Derive the best response functions.

**Solution:** Let the strategy of player 1 be $S_1$ where either $S_1 = L, C$ or $R$. Let the strategy of player 1 be $S_2$ where either $S_2 = T, M$ or $B$. The best response for player for each player is as follows.

$$
\begin{align*}
S_1 &= \begin{cases} 
C & \text{if } S_2 = T \\
L & \text{if } S_2 = M \\
C & \text{if } S_2 = B 
\end{cases} \\
S_2 &= \begin{cases} 
T & \text{if } S_1 = L \\
B & \text{if } S_2 = C \\
B & \text{if } S_1 = R 
\end{cases}
\end{align*}
$$

(b) Are there any dominant strategies?

**Solution:** No. Neither player chooses the same strategy in all cases of their best response function.
Figure 4: A battle of the sexes game for Question 6

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Strategy</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(1,2)</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>(0,0)</td>
<td>(2,1)</td>
<td></td>
</tr>
</tbody>
</table>

(c) Are there any dominated strategies?

**Solution:** Yes. For Player 1, R is dominated by L and C (it never shows up in the best response function. For Player 2, M is dominated by T and B.

(d) Use the method of iterated elimination to try and find the outcome of the game.

**Solution:** If we eliminate M, Player 1’s best response indicates that they will always play S₁ = C. If Player 1 plays S₁ = C, Player 2’s best response is S₂ = B. The outcome is (C, B).

6. Consider the following battle of the sexes game in Figure 4. The payoffs are listed as the (payoff to the player on the left, payoff to the player on top).

(a) List all the Nash Equilibria of this game.

**Solution:** There are two Nash Equilibria: (L, T) and (R, B).

(b) Now change the strategic form game into a sequential choice extensive form game.

**Solution:** Let’s assume that Player 1 chooses first and Player 2 chooses second after observing Player 1’s choice.

(c) Use backwards induction to find the Subgame Perfect Nash Equilibrium.

**Solution:** If Player 1 chooses L, Player 2 will choose T which yields 2 for Player 1. If Player 1 chooses R, Player 2 will choose B which yields 1 for Player 1. The Subgame Perfect Nash Equilibrium is (L, T).
(d) Are the Nash Equilibria you found in Part a still Nash Equilibria of the sequential game?

**Solution:** Yes! The issue isn’t with \((L,T)\), the issue is with \((R,B)\). We can show that \((R,B)\) is still a Nash Equilibrium of the sequential game. Suppose that any player deviates from \((R,B)\) in the sequential game, then they get 0 and are worse off. Thus there is not profitable unilateral deviation from \((R,B)\) in the sequential game.

(e) Are the Nash Equilibria you found in Part a Subgame Perfect Nash Equilibria?

**Solution:** Only \((L,T)\) is a Subgame Perfect Nash Equilibrium.

(f) What’s the difference between a Nash Equilibrium and a Subgame Perfect Nash Equilibrium?

**Solution:** Subgame Perfect Nash Equilibrium require that the strategies be sequentially rational. That is that the strategies would actually be played. We know that when we use backwards induction, we can eliminate \((R,B)\) because Player 1 is always going to choose \(L\). Thus, \(R\) is not sequentially rational.
7. Consider the following strategic form game in Figure 5. The payoffs are listed as the (payoff to the player on the left, payoff to the player on top).

(a) For \( A \geq 0 \) derive the best response functions conditional on the possible values of \( A \).

**Solution:** Let the strategy of player 1 be \( S_1 \) where either \( S_1 = L \) or \( R \). Let the strategy of player 1 be \( S_2 \) where either \( S_2 = T \) or \( B \). The best response for player for each player is as follows.

\[
S_1 = \begin{cases} 
L & \text{if } S_2 = T \text{ and } A < 5 \\
R & \text{if } S_2 = T \text{ and } A > 5 \\
L \text{ or } R & \text{if } S_2 = T \text{ and } A = 5 \\
L & \text{if } S_2 = B 
\end{cases}
\]

\[
S_2 = \begin{cases} 
T & \text{if } S_1 = L \text{ and } A < 5 \\
B & \text{if } S_1 = L \text{ and } A > 5 \\
T \text{ or } B & \text{if } S_1 = L \text{ and } A = 5 \\
T & \text{if } S_1 = R 
\end{cases}
\]

(b) What is the outcome of the game conditional on the possible values of \( A \)?

**Solution:** If \( A \leq 5 \), the outcome is \((L, T)\), these are both weakly dominant strategies. If \( A > 5 \), there is no pure strategy outcome, but there is a mixed strategy Nash equilibrium. The mixed strategy equilibrium has Player 1 playing \( L \) with probability \( \frac{2}{A-3} \) and has Player 2 playing \( T \) with probability \( \frac{2}{A-3} \).

To find the mixed strategy Nash equilibrium, we need to find the probabilities that each player plays a particular strategy. Let \( p_L \) be the probability that...
player 1 plays $L$, and let $p_T$ be the probability that player 2 plays $T$. The probability that player 1 plays $R$ is $p_R = (1 - p_L)$, and the probability that player 2 plays $B$ is $p_B = (1 - p_T)$.

The expected payoffs for each player are as follows.


Each player chooses their respective probability to maximize their expected payoff. Player 1’s problem is as follows.

$$\max_{p_L} E(\pi_1) = p_L p_T 5 + (1 - p_L)p_T A + p_L(1 - p_T)12 + (1 - p_L)(1 - p_T)10$$

The first order condition is

$$\frac{\partial E(\pi_1)}{\partial p_L} = 5p_T - Ap_T + 12(1 - p_T) - 10(1 - p_T)$$

and setting the first order condition equal to zero and solving for $p_T$ yields

$$p_T = \frac{2}{A - 3}.$$ 

If you setup player 2’s problem, you’ll find $p_L$. 