Instructions:
There are 8 questions worth a total of 100 points. Answer each question clearly and concisely. You must show your work to receive credit. You are allowed to work with others, but all work must be your own.

Clearly print your name above and in the space provided on the next page. You must turn in both sides of this cover sheet along with your responses. You do not need to turn in the questions, only your responses with the cover sheet. All pages must be stapled to be graded.
Model Comparison

1. Given an inverse demand function of \( P = 120 - 2Q \) and costs for Firm \( i \) of \( C(q_i) = 20q_i \), find the perfectly competitive equilibrium:

(a) Market Quantity

**Solution:** This is an application of Question 4 taking the limit as \( N \) approaches \( \infty \). From Question 4, \( Q^* = \frac{50N}{(N+1)} \), so \( Q^* = 50 \)

(b) Price

**Solution:** This is an application of Question 4 taking the limit as \( N \) approaches \( \infty \). From Question 4, \( P^* = \frac{120+20N}{(N+1)} \), so \( P^* = 20 \)

(c) Consumer Surplus

**Solution:** This is an application of Question 4 taking the limit as \( N \) approaches \( \infty \). From Question 4, \( CS = \frac{2500N^2}{(N+1)^2} \), so \( CS = 2500 \)

(d) Deadweight Loss

**Solution:** This is an application of Question 4 taking the limit as \( N \) approaches \( \infty \). From Question 4, \( DWL = \frac{2500}{(N+1)^2} \), so \( DWL = 0 \)

(e) If the fixed cost is $10, what is the profit of a single price taking firm?

**Solution:** There are two ways to think about this. If we use the above analysis, a price taking firm sets \( P = MC \) so the equilibrium is described above and the profits are \( \pi^* = -10 \). Instead of using the above analysis, we could say that firms enter as long as it is profitable. In this case, the profit of a single price taking firm would be zero. With this line of thinking we could use the answer from Question 4 part e. Instead of having \( \pi^* = \frac{5000}{(N+1)^2} \), we have \( \pi^* = \frac{5000}{(N+1)^2} - 10 \). Set \( \pi^* = 0 \), and we can solve for \( N \). This gives us the number of firms in a free entry equilibrium and we can use that value of \( N \) \( (N = 21.3607) \) to find the equilibrium primitives of the model.
2. Given an inverse demand function of $P = 120 - 2Q$ and costs for Firm $i$ of $C(q_i) = 20q_i$, find the monopoly equilibrium:

(a) Quantity

Solution: This is an application of Question 4 setting $N = 1$. From Question 4, $Q^* = \frac{50N}{(N+1)}$, so $Q^* = 25$

(b) Price

Solution: This is an application of Question 4 setting $N = 1$. From Question 4, $P^* = \frac{120+20N}{(N+1)}$, so $P^* = 70$

(c) Consumer Surplus

Solution: This is an application of Question 4 setting $N = 1$. From Question 4, $CS = \frac{2500N^2}{(N+1)^2}$, so $CS = 625$

(d) Profit

Solution: This is an application of Question 4 setting $N = 1$. From Question 4, $\pi^* = \frac{5000}{(N+1)^2}$, so $\pi^* = 1250$

(e) Deadweight Loss

Solution: This is an application of Question 4 setting $N = 1$. From Question 4, $DWL = \frac{2500}{(N+1)^2}$, so $DWL = 625$

3. Given an inverse demand function of $P = 120 - 2Q$ and costs for Firm $i$ of $C(q_i) = 20q_i$, find the Cournot duopoly equilibrium:

(a) Firm Quantity

Solution: This is an application of Question 4 setting $N = 2$. From Question 4, $q^* = \frac{50}{(N+1)}$, so $q^* = \frac{50}{3}$

(b) Market Quantity

Solution: This is an application of Question 4 setting $N = 2$. From Question 4, $Q^* = \frac{50N}{(N+1)}$, so $Q^* = \frac{100}{3}$
(c) Price

**Solution:** This is an application of Question 4 setting $N = 2$. From Question 4, 
\[ P^* = \frac{120 + 20N}{(N+1)} \], so \[ P^* = \frac{100}{3} \]

(d) Consumer Surplus

**Solution:** This is an application of Question 4 setting $N = 2$. From Question 4, 
\[ CS = \frac{2500N^2}{(N+1)^2} \], so \[ CS = \frac{10000}{9} \]

(e) Firm Profits

**Solution:** This is an application of Question 4 setting $N = 2$. From Question 4, 
\[ \pi^* = \frac{5000}{(N+1)^2} \], so \[ \pi^* = \frac{5000}{9} \]

(f) Deadweight Loss

**Solution:** This is an application of Question 4 setting $N = 2$. From Question 4, 
\[ DWL = \frac{2500}{(N+1)^2} \], so \[ DWL = \frac{2500}{9} \]

(g) Lerner Index

**Solution:** This is an application of Question 4 setting $N = 2$. From Question 4, the Lerner Index is \[ \frac{100}{120 + 20N} \], so the Lerner Index in this case is \[ \frac{5}{8} \]

(h) HHI

**Solution:** This is an application of Question 4 setting $N = 2$. From Question 4, 
\[ HHI = \frac{1}{N} \], so \[ HHI = \frac{1}{2} \]

(i) What is each firm’s best response function?

**Solution:** This is an application of Question 4 for which the best response function is \[ 4q_i = 100 - 2Q_{-i} \]. Divide by 4 to get \[ q_i = 25 - \frac{1}{2}Q_{-i} \] and now the best response functions are

\[ q_1 = 25 - \frac{1}{2}q_2 \]
\[ q_2 = 25 - \frac{1}{2}q_1 \]

(j) Plot the best response functions in the strategy space.
4. Given an inverse demand function of \( P = 120 - 2Q \) and costs for Firm \( i \) of \( C(q_i) = 20q_i \), find the \( N \) firm symmetric Cournot equilibrium:

(a) Firm Quantity

**Solution:** Let \( Q_{-i} = \sum_{j \neq i}^N q_j \) so that \( Q = q_i + Q_{-i} \)  Setup Firm \( i \)'s problem: choose \( q_i \) to maximize \( \pi_i \)

\[
\begin{align*}
\max_{q_i} \pi_i &= (120 - 2q_i - 2Q_{-i})q_i - 20q_i \\
\frac{d\pi_i}{dq_i} &= 120 - 4q_i - 2Q_{-i} - 20
\end{align*}
\]

set \( \frac{d\pi_i}{dq_i} = 0 \) to get

\[
4q_i = 100 - 2Q_{-i} \quad \text{which is the best response function}
\]

To solve for the equilibrium, impose symmetry conditions. At the symmetric equilibrium, \( q_i = q_j = q^* \) so \( Q = Nq^* \) and \( Q_{-i} = (N - 1)q^* \). The best response function simplifies to

\[
\begin{align*}
4q^* &= 100 - 2(N - 1)q^* \\
2q^* &= 50 - (N - 1)q^* \quad \text{divide by 2} \\
(N + 1)q^* &= 50 \quad \text{add \( (N - 1)q^* \) to each side} \\
q^* &= \frac{50}{(N + 1)}
\end{align*}
\]

(b) Market Quantity

**Solution:** From above \( Q^* = Nq^* \), so \( Q^* = \frac{50N}{(N+1)} \).

(c) Price

**Solution:** The inverse demand function is \( P = 120 - 2Q \), so \( P^* = 120 - 2 \frac{Nq^*}{(N+1)} = \frac{120 + 20N}{(N+1)} \).
(d) Consumer Surplus

**Solution:** Consumer Surplus (CS) is

\[
CS = \int_0^{Q^*} (120 - 2x - P^* dx \\
CS = [(120 - P^*)x - x^2]_0^{Q^*} \\
CS = (120 - P^*)Q^* - Q^*^2 \\
CS = \left( \frac{50N}{(N+1)} \right) \left( \frac{50N}{(N+1)} \right) - \left( \frac{50N}{(N+1)} \right)^2 \\
CS = \frac{5000N^2}{(N+1)^2} - \frac{2500N^2}{(N+1)^2} \\
CS = \frac{2500N^2}{(N+1)^2}
\]

(e) Firm Profits

**Solution:**

\[
\pi^* = P^*q^* - 20q^* \\
\pi^* = (P^* - 20)q^* \\
\pi^* = \left( \frac{120 + 20N}{(N+1)} - \frac{20N + 20}{(N+1)} \right) \frac{50}{(N+1)} \\
\pi^* = \frac{100}{(N+1)} \frac{50}{(N+1)} \\
\pi^* = \frac{5000}{(N+1)^2}
\]
(f) Deadweight Loss

**Solution:** Deadweight Loss (DWL) is

\[
DWL = \int_{Q^*}^{50} 120 - 2x - 20dx
\]

\[
DWL = [(100)x - x^2]_{Q^*}^{50}
\]

\[
DWL = (100)(50 - Q^*) - 2500 + Q^*^2
\]

\[
DWL = 5000 - 2500 + Q^*^2 - 100Q^*
\]

\[
DWL = 2500 - 100Q^* + Q^*^2
\]

\[
DWL = \frac{2500N + 2500}{(N+1)} - \frac{5000N}{(N+1)} + \frac{2500N^2}{(N+1)^2}
\]

\[
DWL = \frac{2500 - 2500N}{(N+1)} + \frac{2500N^2}{(N+1)^2}
\]

\[
DWL = \frac{2500}{(N + 1)^2}((1 - N)(N + 1) + N^2)
\]

\[
DWL = \frac{2500}{(N + 1)^2}
\]

(g) Lerner Index

**Solution:** The Lerner Index is \(\frac{(p^* - MC)}{p^*}\), which in this case is \(\frac{(P^* - 20)}{P^*}\). Now \(P^* - 20 = \frac{120 + 20N - 20N - 20}{(N+1)} = \frac{100}{(N+1)}\). The Lerner Index simplifies to

\[
\frac{(P^* - 20)}{P^*} = \frac{100}{\frac{120 + 20N}{(N+1)}} = \frac{100}{120 + 20N}.
\]

(h) HHI

**Solution:** The \(HHI = \sum_i s_i^2\) where \(s_i\) is the market share of Firm \(i\). Because this is a symmetric model, \(s_i = 1/N\), and the \(HHI = \sum_i \frac{1}{N^2} = \frac{1}{N}\).
5. Given an inverse demand function of \( P = 120 - 2Q \) and costs for Firm \( i \) of \( C(q_i) = 20q_i \), find the duopoly Bertrand equilibrium:

(a) Firm Quantity

**Solution:** With two or more identical firms the Bertrand equilibrium resembles perfect competition. Price competition drives the Bertrand equilibrium price to marginal cost. When \( P^* = 20 \), \( Q^* = 50 \) and assuming that both firms split the market evenly each firm’s quantity is \( q^* = 25 \).

(b) Market Quantity

**Solution:** From above, \( Q^* = 50 \).

(c) Price

**Solution:** From above, \( P^* = 20 \).

(d) Consumer Surplus

**Solution:** Consumers surplus (CS) is \( CS = \int_0^{50} (120 - 2x - 20) \, dx = [100x - x^2]_0^{50} = 100(50) - (50)^2 = 5000 - 2500 = 2500 \).

(e) Firm Profits

**Solution:** Firm profits (\( \pi \)) are \( \pi^* = (P - 20)q = (20 - 20)25 = 0 \).

(f) Deadweight Loss

**Solution:** Deadweight Loss (DWL) is \( DWL = \int_{50}^{50} (120 - 2x - 20) \, dx = 0 \).

(g) Lerner Index

**Solution:** The Lerner Index is \( \frac{P^* - MC}{P^*} = \frac{20 - 20}{20} = 0 \).

(h) HHI

**Solution:** Assuming firms split the market evenly, this question is an application of Question 4 with \( N = 2 \) so the \( HHI = \frac{1}{2} \).
(i) What is each firm’s best response function?

**Solution:** In the Bertrand game, each firm sets their price and the price ranges from $20 \leq P_i \leq 70$. The minimum price any firm would set is their marginal cost and the maximum price results when the firm behaves like a monopolist. Because residual demand is perfectly elastic, profits depend on who offers the lowest price. If Firm 1 offers the lowest price, they serve the entire market. If Firm 2 offers the lowest price, Firm 1 has zero sales. Above we assumed that if both firm offer the same price, they split the market. The profit function for each firm is defined piecewise as follows:

$$
\pi = \begin{cases} 
(120 - 2q_1)q_1 - 20q_1 & \text{if } 20 \leq P_1 < P_2 \leq 70 \\
0 & \text{if } 20 \leq P_2 < P_1 \leq 70
\end{cases}
$$

If Firm 1 has the lowest price, they want to price just below $P_2$ to maximize profits. The best response functions are

$$
P_1 = \begin{cases} 
20 & \text{for } P_2 \leq 20 \\
P_2 - \epsilon & \text{for } 20 < P_2 < 70 \\
70 & \text{for } P_2 \geq 70
\end{cases}
$$

$$
P_2 = \begin{cases} 
20 & \text{for } P_1 \leq 20 \\
P_1 - \epsilon & \text{for } 20 < P_1 < 70 \\
70 & \text{for } P_1 \geq 70
\end{cases}
$$

where $\epsilon$ is positive and sufficiently small.

(j) Plot the best response functions in the strategy space.

6. Compare the results from Questions 1-5:

(a) How does the deadweight loss change across the models?

**Solution:** The deadweight loss in the Bertrand model is zero. To examine the rest of the models we only need to look at Question 4. In Question 4, the deadweight loss (DWL) is $DWL = \frac{2500}{(N+1)^2}$. Now $\frac{d(DWL)}{dN} = -\frac{2500(2)(N+1)}{(N+1)^3}$ so that as $N$ increases, the deadweight loss decreases.

(b) How does the Lerner Index change across the models?

**Solution:** The Lerner Index in the Bertrand model is zero. To examine the rest of the models we only need to look at Question 4. In Question 4, the Lerner Index is $\frac{100}{120+20N}$. Now $\frac{d(LernerIndex)}{dN} = -\frac{100(20)}{(120+20N)^2}$ so that as $N$ increases, the Lerner Index decreases.

(c) How are the duopoly Cournot and Bertrand best response functions different?
Solution: The slope of the best response function differs. For the Cournot duopoly, the slope is negative so the goods are strategic substitutes. For the Bertrand duopoly, the slope is positive so the goods are strategic complements.

Cournot Model with Asymmetric Costs

7. The inverse demand function is \( P = 120 - 2Q \), the costs for Firm 1 are \( C(q_1) = 12q_1 \) and the costs for Firm 2 are \( C(q_2) = c_2q_2 \) where \( c_2 \geq 0 \) is a parameter. Assume firms compete in quantities.

(a) Find the best response functions for each firm.

Solution: Start with Firm 2’s problem: setup the profit equation, differentiate with respect to \( q_2 \), set the first order condition equal to zero and solve for the best response function.

\[
\pi_2 = 120q_2 - 2q_2^2 - 2q_2q_1 - c_2q_2
\]

\[
\frac{d\pi_2}{dq_2} = 120 - 4q_2 - 2q_1 - c_2
\]

\[
0 = 120 - 4q_2 - 2q_1 - c_2
\]

\[
4q_2 = 120 - 2q_1 - c_2
\]

\[
q_2 = 30 - \frac{1}{2}q_1 - \frac{1}{4}c_2
\]

This is Firm 2’s best response function. Firm 1’s best response function is found in a similar way or you can alternate the \( q_2 \)’s and \( q_1 \)’s and set \( c_2 = 12 \) in Firm 2’s best response function to get

\[
q_1 = 30 - \frac{q_2}{2} - \frac{12}{4} = 27 - \frac{q_2}{2}
\]

which is Firm 1’s best response function.
(b) Find the equilibrium quantities for each firm as a function of \( c_2 \).

**Solution:** Find the equilibrium by the intersection of the best response functions.

\[
q_1 = 27 - \frac{1}{2}(30 - \frac{1}{2}q_1 - \frac{1}{4}c_2) \\
q_1 = 27 - 15 + \frac{1}{4}q_1 + \frac{1}{8}c_2 \\
3q_1 = 12 + \frac{1}{8}c_2 \\
q_1^* = 16 + \frac{1}{6}c_2
\]

Firm 2’s equilibrium quantity is

\[
q_2 = 30 - \frac{1}{2}q_1 - \frac{1}{4}c_2 \\
q_2 = 30 - \frac{1}{2}(16 + \frac{1}{6}c_2) - \frac{1}{4}c_2 \\
q_2 = 30 - 8 - \frac{1}{12}c_2 - \frac{1}{4}c_2 \\
q_2^* = 22 - \frac{1}{3}c_2
\]

(c) Comparative Statics: Find \( \frac{dq_i^*}{dc_2} \) for \( i = 1, 2 \).

**Solution:**

\[
\frac{dq_2^*}{dc_2} = -\frac{1}{3} \\
\frac{dq_1^*}{dc_2} = \frac{1}{6}
\]

(d) Describes what happens to \( q_1^* \) and \( q_2^* \) as \( c_2 \) increases.

**Solution:** \( \frac{dq_1^*}{dc_2} > 0 \) so as \( c_2 \) increases \( q_1^* \) increases. \( \frac{dq_2^*}{dc_2} < 0 \) so as \( c_2 \) increases \( q_2^* \) decreases.

(e) What happens to the market quantity as \( c_2 \) increases.

**Solution:** Market quantity is

\[
Q^* = q_1^* + q_2^* = 38 - \frac{1}{6}c_2 \\
\frac{dQ^*}{dc_2} = \frac{dq_1^*}{dc_2} + \frac{dq_2^*}{dc_2} = -\frac{1}{6} < 0
\]

As \( c_2 \) increases market quantity decreases.

(f) Is there a value of \( c_2 \) such that \( q_2^* = 0 \)? If so, what is this value. If not, why?
8. The inverse demand function is $P = 120 - 2Q$, the costs for Firm 1 are $C(q_1) = 12q_1$ and the costs for Firm 2 are $C(q_2) = 18q_2$. Assume firms compete in quantities.

(a) Find the best response functions for each firm.

**Solution:** This is an application of the previous question with $c_2 = 18$. From the previous question, $q_2^* = 22 - \frac{1}{3}c_2$ and $q_1^* = 27 - \frac{q_2^*}{2}$. The best response functions are

\[
q_1 = 27 - \frac{q_2}{2} \\
q_2 = \frac{51}{2} - \frac{q_1}{2}
\]

(b) Find the equilibrium quantities for each firm.

**Solution:** This is an application of the previous question with $c_2 = 18$. From the previous question, $q_2^* = 22 - \frac{1}{3}c_2$ and $q_1^* = 16 + \frac{1}{6}c_2$. The equilibrium quantities are

\[
q_1^* = 19 \\
q_2^* = 16
\]

(c) What is the equilibrium price?

**Solution:** The equilibrium price is $P^* = 120 - 2Q^* = 120 - 2(35) = 50$. 

(d) Find the consumers surplus.

Solution: Consumer surplus (CS) is

\[ CS = \int_0^{Q^*} 120 - 2x - P^* \, dx \]

\[ CS = [(120 - P^*)x - x^2]_0^{Q^*} \]

\[ CS = (120 - P^*)Q^* - Q^*^2 \]

\[ CS = (120 - 50)35 - 35^2 \]

\[ CS = 35^2 = 1225 \]

(e) What are each firm’s profits?

Solution: For Firm 1:

\[ \pi_1^* = P^*q_1^* - 12q_1^* \]

\[ = 50(19) - 12(19) \]

\[ = 722 \]

For Firm 2:

\[ \pi_2^* = P^*q_2^* - 18q_2^* \]

\[ = 50(16) - 18(16) \]

\[ = 512 \]