Instructions:
There are 2 questions worth a total of 100 points. Answer each question clearly and concisely. You must show your work to receive credit. You are allowed to work with others, but all work must be your own.

Clearly print your name above and in the space provided on the next page. You must turn in both sides of this cover sheet along with your responses. You do not need to turn in the questions, only your responses with the cover sheet. All pages must be stapled to be graded.
Limit Pricing

1. The inverse of market demand is \( P = 120 - 4Q \). There are two firms, an incumbent and a potential entrant, and both firms compete in quantities. The costs of each firm are \( C(q) = 20q \) and the potential entrant faces an entry cost of \( E^2 \). The interest rate is 25%. Assume that the incumbent can costlessly and credibly commit to any production level. Answer the following questions.

(a) What is the incumbent’s limit price as a function of \( E \)?

**Solution:** Let \( A \) be the incumbent and \( B \) be the potential entrant. The profit maximization problem for firm \( B \) is as follows.

\[
\max_{q_B} \pi_B = \left( \frac{1}{0.25} \right) \left( (120 - 4q_A - 4q_B)q_B - 20q_B \right) - E^2
\]

\[
\frac{d\pi_B}{dq_B} = \left( \frac{1}{0.25} \right) (120 - 4q_A - 8q_B - 20)
\]

\[
0 = 100 - 4q_A - 8q_B
\]

\[
q_B = \frac{25}{2} - \frac{q_A}{2}
\]

Now substitute \( q_B \) back into \( \pi_B \) and solve for \( q_A \) such that \( \pi_B = 0 \).

\[
\pi_B = \left( \frac{1}{0.25} \right) \left( (120 - 4q_A - 4q_B)q_B - 20q_B \right) - E^2
\]

\[
\pi_B = 4 \left( (100 - 4q_A - (50 - 2q_A)) \left( \frac{25}{2} - \frac{q_A}{2} \right) \right) - E^2
\]

\[
\pi_B = (50 - 2q_A)(50 - 2q_A) - E^2
\]

\[
0 = (50 - 2q_A)^2 - E^2
\]

\[
E = 50 - 2q_A
\]

\[
q_A^L = 25 - \frac{E}{2}
\]

(b) What’s the level of entry cost, \( E \), such that the incumbent is indifferent to accommodating entry and using a limit pricing strategy?

**Solution:** If firm A accommodates entry, we can find accommodation profits using firm B’s best response above and imposing symmetry.

\[
q_B = \frac{25}{2} - \frac{q_A}{2}
\]

\[
q^* = \frac{25}{2} - \frac{q^*}{2}
\]

\[
q^* = \frac{25}{3}
\]
With accommodation, the market quantity is \( Q = \frac{50}{3} \) and the market price is \( P = 120 - 4\frac{50}{3} = 100 \frac{2}{3} = 160 \frac{4}{3} \). The discounted sum of accommodation profits are
\[
\pi^A = 4\left( \frac{100}{3} \left( \frac{25}{3} \right) - 20 \left( \frac{25}{3} \right) \right) = \frac{10000}{9}
\]
If firm A uses a limit pricing strategy to deter entry, the discounted sum of profits are as follows.
\[
\begin{align*}
\pi^L_A &= 4(100 - 4q_A^L)q_A^L \\
\pi^L_A &= 4(100 - 4(25 - E))(25 - E) \\
\pi^L_A &= 4(2E)(25 - E) \\
\pi^L_A &= 200E - 4E^2 \\
\end{align*}
\]
The question asks to find \( E \) such that \( \frac{10000}{9} = 200E - 4E^2 \). Rearranging provides the following quadratic equation: \( E^2 - 50E + \frac{2500}{9} = 0 \). Solving the quadratic equation, we find that if \( E = 25 - \frac{25}{3}\sqrt{5} \), the incumbent is indifferent to accommodating entry and using a limit pricing strategy.

(c) If the incumbent didn’t face a potential entrant, what quantity would they set and what is the discounted sum of profits?

**Solution:** If the incumbent, firm A, is a monopolist their profit maximization problem is as follows.
\[
\begin{align*}
\max q_A \pi_A &= \left( \frac{1}{0.25} \right) \left( (120 - 4q_A)q_A - 20q_A \right) \\
d\pi_A \left/ dq_A \right. &= (4) (120 - 8q_A - 20) \\
0 &= 100 - 8q_A \\
q_A &= \frac{25}{2}
\end{align*}
\]
Substitute \( q_A \) back into the profit equation to get \( \pi_A = 2500 \).

(d) If \( E = 26 \), what is the limit priced used by the firm? (Note: this is a trick to this question.)

**Solution:** From above \( q_A^L = 25 - \frac{E}{2} \), so if \( E = 26 \), \( q_A^L = 12 \). Note that the monopoly quantity above is \( \frac{25}{2} \). If the firm’s quantity is either 12 or 12.5, entry is deterred. (A higher quantity translates to a lower price so entry is less likely.) Because both 12 and 12.5 deter entry, think about what gives the firm greater profits. A quantity of 12.5 is determined from the monopoly solution so profits are greater when the quantity is 12.5. When the quantity is 12.5, entry is
Strategic Bundling

2. There are two firms, A and B, with a constant marginal cost of $10. There are two markets, 1 and 2, with different goods. Firm A has a monopoly in market 1 so \( q_1 = q_{1A} \), and both firms compete in quantities in the second market so \( Q_2 = q_{2A} + q_{2B} \). There is a representative consumer who’s utility maximization problem is as follows.

\[
\max_{m, Q_1, Q_2} U = m + 100Q_1 - \frac{1}{2}Q_1^2 + 100Q_2 - \frac{1}{2}Q_2^2
\]

subject to

\[
Y = p_1 Q_1 + p_2 Q_2 + m
\]

\( m \) is a composite good with a price normalized to 1. \( Y \) is the representative consumer’s income and \( p_1 \) and \( p_2 \) are the prices of their respective goods.

(a) Find the profits of each firm.

**Solution:** Setup the Lagrangian for the representative consumer and find the demand for each good.

\[
\max_{m, Q_1, Q_2} \mathcal{L} = m + 100Q_1 - \frac{1}{2}Q_1^2 + 100Q_2 - \frac{1}{2}Q_2^2 + \lambda(Y - p_1 Q_1 - p_2 Q_2 - m)
\]

\[
\frac{d\mathcal{L}}{dm} = 1 - \lambda
\]

\[
\frac{d\mathcal{L}}{dQ_1} = 100 - Q_1 - \lambda p_1
\]

\[
\frac{d\mathcal{L}}{dQ_2} = 100 - Q_2 - \lambda p_2
\]

\[
0 = 1 - \lambda
\]

\[
0 = 100 - Q_1 - \lambda p_1
\]

\[
0 = 100 - Q_2 - \lambda p_2
\]

\[
\lambda = 1
\]

\[
p_1 = 100 - Q_1
\]

\[
p_2 = 100 - Q_2
\]

Firm A’s profit maximization problem with respect to the first market is as
max $\pi_{1A} = (100 - q_{1A})q_{1A} - 10q_{1A}$
\[ \frac{d\pi_{1A}}{dq_{1A}} = 100 - 10 - 2q_{1A} \]
\[ 0 = 90 - 2q_{1A} \]
\[ q_{1A} = 45 \]
\[ \pi_{1A} = 2025 \]

Setup firm A’s profit maximization problem with respect to the second market and then impose symmetry to find the equilibrium for both firms.

max $\pi_{2A} = (100 - q_{2A} - q_{2B})q_{2A} - 10q_{2A}$
\[ \frac{d\pi_{2A}}{dq_{2A}} = 100 - q_{2B} - 10 - 2q_{1A} \]
\[ 0 = 90 - q_{2B} - 2q_{2A} \]
\[ 0 = 90 - q_{2}^* - 2q_{2}^* \]
\[ q_{2}^* = 30 \]
\[ \pi_{2A} = \pi_{2B} = 900 \]

The profits for firm A are $\pi_A = \pi_{1A} + \pi_{2A} = 2925$ and the profits for firm B are $\pi_B = \pi_{2B} = 900$.

(b) Find the profits of each firm when firm A offers a pure bundle of $b_A = q_{1A} + q_{2A}$. Is firm A better off?

**Solution:** The pure bundle from firm A changes the representative consumer’s decision because now when they choose good 1 their forced to also purchase good 2. The relevant prices are now the price of the bundle, $p_A$, the price of the second good, $p_B$ (since it’s only offered from firm B), and the price of the composite good which is normalized to 1. Market quantity of the first good is $Q_1 = b_A$ and the market quantity of the second good is $Q_2 = b_A + q_{2B}$.

Using this information, setup the Lagrangian for the representative consumer
and find the demand for each good.

$$\max_{m,b_A,q_{2B}} L = m + 100b_A - \frac{1}{2}b_A^2 + 100(b_A + q_{2B}) - \frac{1}{2}(b_A + q_{2B})^2$$

$$+ \lambda(Y - p_Ab_A - p_Bq_{2B} - m)$$

$$\frac{dL}{dm} = 1 - \lambda$$

$$\frac{dL}{db_A} = 100 - b_A + 100 - (b_A + q_{2B}) - \lambda p_A$$

$$\frac{dL}{dq_{2B}} = 100 - (b_A + q_{2B}) - \lambda p_B$$

$$0 = 1 - \lambda$$

$$0 = 200 - 2b_A - q_{2B} - \lambda p_A$$

$$0 = 100 - b_A - q_{2B} - \lambda p_B$$

$$\lambda = 1$$

$$p_A = 200 - 2b_A - q_{2B}$$

$$p_B = 100 - b_A - q_{2B}$$

Now setup each firm’s profit maximization problem.

$$\max_{b_A} \pi_A = (200 - 2b_A - q_{2B})b_A - 20b_A$$

$$\frac{d\pi_A}{db_A} = 200 - q_{2B} - 4b_A - 20$$

$$0 = 180 - q_{2B} - 4b_A$$

$$b_A = 45 - \frac{q_{2B}}{4}$$

$$\max_{q_{2B}} \pi_B = (100 - b_A - q_{2B})q_{2B} - 10q_{2B}$$

$$\frac{d\pi_B}{dq_{2B}} = 100 - 2q_{2B} - b_A - 10$$

$$0 = 90 - 2q_{2B} - b_A$$

$$q_{2B} = 45 - \frac{b_A}{2}$$
Substitute one best response function into the other to solve for \( b_A \) and \( q_{2B} \).

\[
4b_A = 180 - q_{2B} \\
4b_A = 180 - 45 + \frac{b_A}{2} \\
\frac{7}{2}b_A = 135 \\
b_A = \frac{270}{7} \\
q_{2B} = \frac{180}{7}
\]

Now substitute the quantities back into the profit equations to find profits.

\[
\pi_A = (180 - 2b_A - q_{2B})b_A = \frac{145800}{49} = 2975.51 \\
\pi_B = (90 - b_A - q_{2B})q_{2B} = \left(\frac{180}{7}\right)^2 = 661.22
\]

Firm A is better off with the pure bundle since 2975.5 > 2925.

(c) Find the profits of each firm when firm A offers a mixed bundle where firm A offers two products: \( b_A \) and \( q_{1A} \) where \( b_A = q_{1A} + q_{2A} \). (Note that firm A does not offer \( q_{2A} \) separately in this mixed bundle.) Is firm A better off with no bundling, a pure bundle or the mixed bundle?

**Solution:** With the mixed bundle, there are four prices: the price of the bundle \( p_A \), the price of good 1 \( p_1 \), the price of good 2 \( p_2 \), and the price of the composite good which is normalized to 1. Market quantity of the first good is \( Q_1 = b_A + q_{1A} \) and the market quantity of the second good is \( Q_2 = b_A + q_{2B} \).

Using this information, setup the Lagrangian for the representative consumer.
and find the demand for each good.

\[
\max_{m, b_A, q_{1A}, q_{2B}} \mathcal{L} = m + 100(b_A + q_{1A}) - \frac{1}{2}(b_A + q_{1A})^2 + 100(b_A + q_{2B}) - \frac{1}{2}(b_A + q_{2B})^2 + \lambda(Y - p_A b_A - p_1 q_{1A} - p_B q_{2B} - m)
\]

\[
\begin{align*}
\frac{d\mathcal{L}}{dm} &= 1 - \lambda \\
\frac{d\mathcal{L}}{db_A} &= 100 - b_A - q_{1A} + 100 - b_A - q_{2B} - \lambda p_A \\
\frac{d\mathcal{L}}{dq_{1A}} &= 100 - b_A - q_{1A} - \lambda p_1 \\
\frac{d\mathcal{L}}{dq_{2B}} &= 100 - b_A - q_{2B} - \lambda p_B
\end{align*}
\]

\[
\begin{align*}
0 &= 1 - \lambda \\
0 &= 200 - 2b_A - q_{1A} - q_{2B} - \lambda p_A \\
0 &= 100 - b_A - q_{1A} - \lambda p_1 \\
0 &= 100 - b_A - q_{2B} - \lambda p_B \\
\lambda &= 1 \\
p_A &= 200 - 2b_A - q_{1A} - q_{2B} \\
p_1 &= 100 - b_A - q_{1A} \\
p_B &= 100 - b_A - q_{2B}
\end{align*}
\]

Now setup each firm’s profit maximization problem.

\[
\max_{b_A, q_{1A}} \pi_A = (200 - 2b_A - q_{1A} - q_{2B})b_A - 20b_A + (100 - b_A - q_{1A})q_{1A} - 10q_{1A}
\]

\[
\begin{align*}
\frac{d\pi_A}{db_A} &= 200 - q_{1A} - q_{2B} - 4b_A - 20 - q_{1A} \\
\frac{d\pi_A}{dq_{1A}} &= -b_A + 100 - b_A - 2q_{1A} - 10 \\
0 &= 180 - 2q_{1A} - q_{2B} - 4b_A \\
0 &= 90 - 2b_A - 2q_{1A} \\
q_{1A} &= 45 - b_A \\
b_A &= 45 - \frac{q_{1A}}{2} - \frac{q_{2B}}{4} \\
b_A &= \frac{45}{2} + \frac{b_A}{2} - \frac{q_{2B}}{4} \\
b_A &= 45 - \frac{q_{2B}}{2}
\end{align*}
\]
\[
\max_{q_{2B}} \pi_B = (100 - b_A - q_{2B})q_{2B} - 10q_{2B}
\]
\[
\frac{d\pi_B}{dq_{2B}} = 100 - 2q_{2B} - b_A - 10
\]
\[
0 = 90 - 2q_{2B} - b_A
\]
\[
q_{2B} = 45 - \frac{b_A}{2}
\]

Substitute one best response function into the other to solve for \(b_A\) and \(q_{2B}\).

\[
2b_A = 90 - q_{2B}
\]
\[
2b_A = 90 - 45 + \frac{b_A}{2}
\]
\[
\frac{3}{2}b_A = 45
\]
\[
b_A = 30
\]
\[
q_{2B} = 30
\]

Using one of the best response functions above, \(q_{1A} = 45 - b_A = 15\).

Now substitute the quantities back into the profit equations to find profits.

\[
\pi_A = (180 - 2b_A - q_{1A} - q_{2B})b_A + (90 - b_A - q_{1A})q_{1A}
\]
\[
\pi_A = (180 - 105)30 + (45)15 = 2925
\]
\[
\pi_B = (90 - b_A - q_{2B})q_{2B} = 900
\]

Firm A is better off with the pure bundle since \(2975.5 > 2925\).