EE 477
Digital Signal Processing

3

Frequency Spectrum

Sums of Sinusoids

- We have seen that adding two sinusoids with the same frequency results in another sinusoid with the same frequency.
- Consider adding sinusoids with different frequencies:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

Sum in Phasor Form

- Can also express sum as:

$$x(t) = A_0 + \sum_{k=1}^{N} \Re e\left\{ \underbrace{A_k e^{j\phi_k}}_{phasor\ X_k} e^{j2\pi f_k t} \right\}$$

- Or via Euler:

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

Positive and Negative Freqs

- Interpret sinusoidal sum as two-sided, with pairs of rotating phasors, one positive frequency f_k and one negative frequency $-f_k$

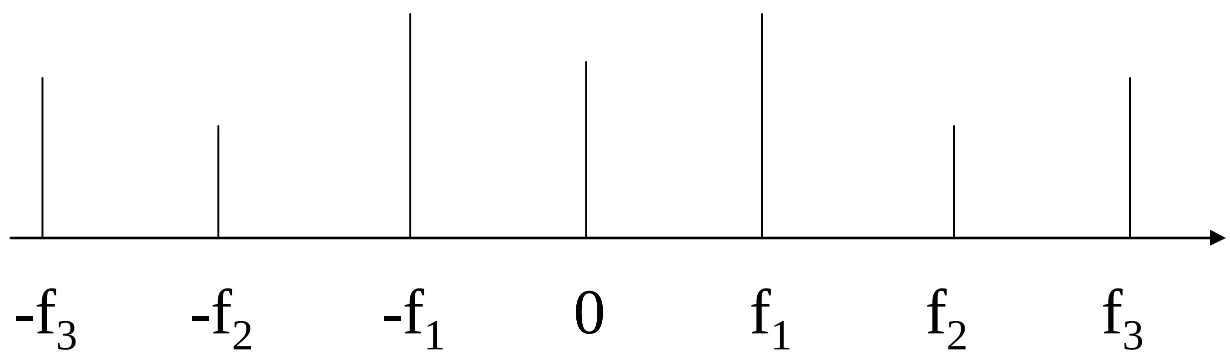
$$\frac{X_k}{2} e^{j2\pi f_k t} \text{ rotates counter clockwise}$$

$$\frac{X_k^*}{2} e^{-j2\pi f_k t} \text{ rotates clockwise}$$

Frequency Domain Representation

- Represent x(t) in *frequency domain* using mag&phase @ *f*:

$$\left(X_0,0\right),\left(\tfrac{1}{2}X_1,f_1\right),\left(\tfrac{1}{2}X_1^*,-f_1\right),\left(\tfrac{1}{2}X_2,f_2\right),\left(\tfrac{1}{2}X_2^*,-f_2\right),\ldots$$



Products of Sinusoids

- The sum of two sinusoids contains only those two sinusoidal frequencies. What about *multiplying* two sinusoids?

$$x(t) = \cos(\omega_0 t)\cos(\omega_1 t)$$
$$= \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)\left(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\right)$$
$$= \left(\frac{e^{j(\omega_0+\omega_1)t} + e^{-j(\omega_0+\omega_1)t} + e^{j(\omega_0-\omega_1)t} + e^{-j(\omega_0-\omega_1)t}}{4}\right)$$
$$= \left(\frac{e^{j(\omega_0+\omega_1)t} + e^{-j(\omega_0+\omega_1)t}}{4}\right) + \left(\frac{e^{j(\omega_0-\omega_1)t} + e^{-j(\omega_0-\omega_1)t}}{4}\right)$$
$$= \tfrac{1}{2}\cos(\omega_0 + \omega_1)t + \tfrac{1}{2}\cos(\omega_0 - \omega_1)t$$

Product (cont.)

- Note that the product can be expressed as *frequency sum* and *frequency difference* components.
- Or conversely, a pair of frequency components can be expressed as a *product*, as in amplitude modulation.

Periodic Waveforms

- Periodic complicated waveforms can be expressed as *harmonic sums*.

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi \cdot kf_0 \cdot t + \phi_k), \quad kf_0 = f_k$$

- The *period* of the signal is $T_0 = 1/f_0$. This is called the *fundamental frequency* or *fundamental period*.

Fourier Analysis

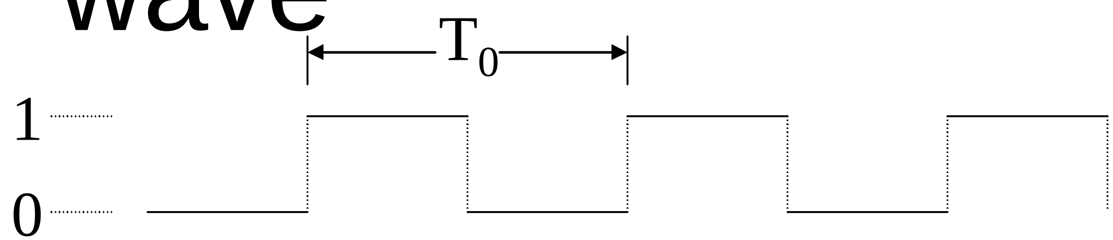
- What if we have a periodic signal and we want to figure out the X_k values (magnitude and phase)?

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t) e^{-j2\pi kt/T_0}\,dt, \quad k = 1,2,\ldots$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

Fourier example: square wave

$$x(t) = \begin{cases} 1, & 0 \le t < T_0/2 \\ 0, & T_0/2 \le t < T_0 \end{cases}$$



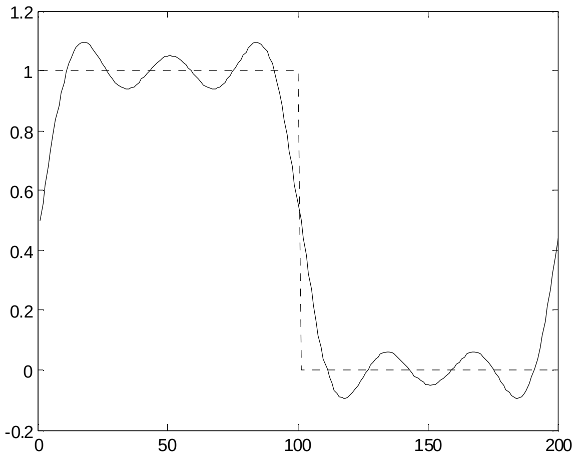
$$X_k = \frac{2}{T_0}\int_0^{T_0/2} 1 e^{-j2\pi kt/T_0}\,dt; \text{ and for } k = 0 \quad X_0 = \frac{1}{T_0}\int_0^{T_0/2} 1\,dt = \frac{1}{2}$$

$$= \frac{2}{T_0}\left(\frac{e^{-j2\pi kt/T_0}}{-j2\pi k/T_0}\right)\Bigg|_0^{T_0/2}$$

$$= \frac{e^{-j\pi k} - 1}{-j\pi k} = \frac{1 - e^{-j\pi k}}{j\pi k} = \begin{cases} 0, & k \text{ even} \\ \dfrac{2}{j\pi k}, & k \text{ odd} \end{cases}$$

Square wave (cont.)

- Note: only odd harmonics are present
- Note: harmonics decline as 1/k
- Note: phase from 1/j = -j implies $-\pi/2$, and $\sin(\theta) = \cos(\theta-\pi/2)$



Time-varying Amp and Freq

- What if we allow amplitude and frequency to vary as functions of time?

$$x(t) = A_0(t) + \sum_{k=1}^{N} A_k(t)\cos(\psi(t))$$

- The *instantaneous frequency* is the time derivative of the phase function $\psi(t)$:

$$\omega_i(t) = \frac{d}{dt}\psi(t)$$

Time varying (cont.)

- Instantaneous frequency is the slope of the phase function
- Example: constant frequency

$$\omega_i(t) = \frac{d}{dt}\left(\omega_0 t + \phi\right) = \omega_0$$

- Example: linearly increasing frequency

$$\omega_i(t) = \frac{d}{dt}\left(\frac{a}{2}\omega_0 t^2 + \omega_0 t + \phi\right) = \omega_0(1 + a \cdot t)$$

Time Varying (cont.)

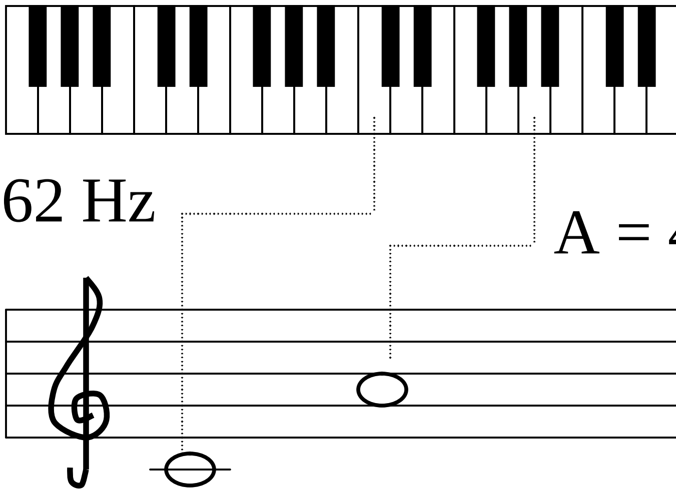
- Since amplitude and frequency vary with time, we want to estimate *short-time* spectrum.
- Concept: perform a series of Fourier "snap shots" for short segments of the signal
- This is known as a short-time Fourier transform, or a *spectrogram*

An aside: musical frequencies

- Music is often based on harmonic signals with nice “consonant” relationships
- Western music uses an *octave* (factor of 2) basis with a *scale* of 12 notes per octave.
- Modern music has an equal-tempered scale such that adjacent notes have the same frequency ratio: $r = 2^{1/12} = 1.059463$

  (note m in the scale has $f_m = f_0 * 2^{m/12}$)

Musical Scale



$C = 440(2^{-9/12}) = 261.62$ Hz

A = 440 Hz = reference