EELE 477
Digital Signal Processing

8b

IIR Systems

Inverse z-Transform

- Since $Y(z) = H(z)\,X(z)$, we can use the system function $H(z)$ and the input $X(z)$ to find the output transform $Y(z)$
- If we have $Y(z)$, then how should we get $y[n]$??
- Usual way: manipulate Y(z) into a sum of recognizable terms from a transform table

Inverse Transform Tables

- Essential transform pairs:

$$x[n-n_0] \Leftrightarrow z^{-n_0} X(z)$$

$$\delta[n] \Leftrightarrow 1$$

$$\delta[n-n_0] \Leftrightarrow z^{-n_0}$$

$$u[n] \Leftrightarrow \frac{1}{1-z^{-1}}$$

$$a^n u[n] \Leftrightarrow \frac{1}{1-az^{-1}}$$

Partial Fraction Expansion

- Separate ratio of polynomials into a sum of simpler polynomial ratios, then “take the inverse transform” by matching to the table
- Ex:

$$\frac{b_0+b_1z^{-1}}{\left(1-a_1z^{-1}\right)\left(1-z^{-1}\right)}=\frac{\overset{\overset{Aa_1^n u[n]}{\nearrow}}{A}}{1-a_1z^{-1}}+\frac{\overset{\overset{B\,u[n]}{\nearrow}}{B}}{1-z^{-1}}$$

$$=\frac{A+B-\left(A+Ba_1\right)z^{-1}}{\left(1-a_1z^{-1}\right)\left(1-z^{-1}\right)}$$

Partial Fraction (cont.)

- Equate A and B terms with numerator

$$A + B = b_0$$

$$A + Ba_1 = -b_1$$

- Solving:

$$B = \frac{b_0 + b_1}{1 - a_1}$$

$$A = \frac{b_0 a_1 + b_1}{a_1 - 1}$$

Partial Fraction (cont.)

- Alternative: evaluate by isolating the coefficients

$$H(z)=\frac{b_0+b_1z^{-1}}{\left(1-a_1z^{-1}\right)\left(1-z^{-1}\right)}=\frac{A}{1-a_1z^{-1}}+\frac{B}{1-z^{-1}}$$

$$H(z)\left(1-a_1z^{-1}\right)=\frac{b_0+b_1z^{-1}}{\left(1-z^{-1}\right)}=A+\cancelto{0\ (z=a_1)}{\frac{B\left(1-a_1z^{-1}\right)}{1-z^{-1}}}$$

$$H(z)\left(1-a_1z^{-1}\right)\Big|_{z=a_1}=\frac{b_0+b_1a_1^{-1}}{\left(1-a_1^{-1}\right)}=A$$

Partial Fraction (cont.)

- This approach works fine for distinct poles. If duplicate poles, need to use a similar systematic procedure (see an advanced DSP text, or take EELE 577)

Second-order System

- Typically implement *higher-order* IIR systems using *second-order* blocks
- Why?
  - Helps isolate coefficients to lower quantization sensitivity
  - Each block implements a pair of complex conjugate poles (or real)

Second-order Response

- General second order:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

$$= -\frac{b_2}{a_2} + \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

- Unit sample response form:

$$h[n] = -\left(\frac{b_2}{a_2}\right)\delta[n] + A_1\left(p_1\right)^n u[n] + A_2\left(p_2\right)^n u[n]$$

2^{nd}-order poles: real

- The two poles of the 2^{nd}-order system are either both real, or are complex conjugates
- If p_1 and p_2 are real, impulse response is two decaying exponentials of the form p_1^n and p_2^n (e.g., $\left(\frac{1}{3}\right)^n u[n]$ or $\left(-\frac{1}{8}\right)^n u[n]$

2nd-order poles: complex

- If p_1 and p_2 are complex conjugates, express in polar form as $p_1 = re^{j\theta}$

$$p_2 = re^{-j\theta} = p_1^*$$

$$h[n] = -\left(\frac{b_2}{a_1}\right)\delta[n] + A_1\left(re^{j\theta}\right)^n u[n] + A_2\left(re^{-j\theta}\right)^n u[n]$$

$$= -\left(\frac{b_2}{a_1}\right)\delta[n] + A_1 r^n e^{j\theta n} u[n] + A_2 r^n e^{-j\theta n} u[n]$$

Complex poles (cont.)

- Denominator:

$$
\begin{aligned}
1 - a_1 z^{-1} - a_2 z^{-2} &= \left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right) \\
&= \left(1 - re^{j\theta} z^{-1}\right)\left(1 - re^{-j\theta} z^{-1}\right) \\
&= 1 - \underbrace{2r\cos\theta}_{a_1} z^{-1} - \underbrace{r^2}_{a_2} z^{-2}
\end{aligned}
$$

- Note influence of pole locations on coefficients a_1 and a_2